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## Is the Tsallis entropy stable?

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**Abstract** – The question of whether the Tsallis entropy is Lesche-stable is revisited. It is argued that when physical averages are computed with the escort probabilities, the correct application of the concept of Lesche-stability requires use of the escort probabilities. As a consequence, as shown here, the Tsallis entropy is unstable but the thermodynamic averages are stable. We further show that Lesche stability as well as thermodynamic stability can be obtained if the homogeneous entropy is used as the basis of the formulation of non-extensive thermodynamics. In this approach, the escort distribution arises naturally as a secondary structure.

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**Introduction.** – The concept of non-extensive thermodynamics was introduced by Tsallis about 20 years ago [1] and has generated a large literature. The original idea was that for systems out of equilibrium where the Boltzmann distribution no longer holds, the Boltzmann entropy could be replaced by a more general function while maintaining the formalism of thermodynamics. In particular, maximization of the entropy under the usual constraints (normalized probabilities, fixed internal energy) yields the so-called  $q$ -exponential distribution that generalizes the usual Boltzmann distribution of classical statistical mechanics. While this seems straightforward now, it did in fact require considerable effort to arrive at the now-accepted form of the theory. A particular issue that was historically important and that remains problematic is the notion of thermodynamic stability since the Tsallis entropy gives negative specific heats in certain circumstances [2–5]. Nevertheless, this was one of the issues that motivated the advocacy of nontrivial averaging procedures in non-extensive thermodynamics [3].

A new controversy has arisen based on a recent paper by Abe where it is shown that averages computed within the non-extensive formalism are unstable in the sense that a small change in the distribution function can lead to a large change in the computed average [6]. This surprising result should be understood in a broader context wherein it was originally asked whether the Tsallis entropy is

stable with respect to changes in the distribution. This was shown to be true by Abe [7] so the result that averages of observables are unstable while the entropy is stable appears quite surprising. In this paper we question whether either fact has actually been proven. In short, our argument is that Lesche-stability is motivated by making correspondence with an experimental procedure and that this means it should be understood in terms of the probabilities that govern the observation of a given microstate. In the usual formulation of non-extensive thermodynamics, those are the escort probabilities. When understood in this way, it is easy to show that the Tsallis entropy is not Lesche-stable.

This would appear to create an uncomfortable situation in which the Tsallis formulation is not Lesche-stable and in which thermodynamic stability is also problematic. We contend that this can be resolved by a shift of viewpoint in which the physical probabilities are taken as being fundamental. While the Tsallis entropy cannot be satisfactorily formulated in this way [8], a closely related functional, the homogeneous entropy, appears as a natural alternative. We show that the homogeneous entropy is in fact Lesche stable, gives positive-definite specific heats, yields the usual  $q$ -exponential distributions when maximized and gives rise to a consistent thermodynamics.

**Non-extensive thermodynamics.** –

*Tsallis formalism.* The usual non-extensive formalism can be illustrated as follows. Consider a system composed

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of some number,  $n$ , of microstates and let  $p_i$  be the probability associated with the  $i$ -th microstate. The Tsallis entropy is computed as

$$S_q = \frac{1 - \sum_{i=1}^n p_i^q}{q-1}, \quad (1)$$

where  $q$  is the index characterizing the entropy functional. The limit  $q \rightarrow 1$  gives the usual Boltzmann entropy. One subtlety in the theory of non-extensive thermodynamics is that the average of an observable,  $\mathcal{O}$ , that takes the value  $O_i$  in the  $i$ -th state is evaluated using the so-called escort distribution [9] giving

$$\langle \mathcal{O} \rangle = \sum_{i=1}^n \frac{p_i^q}{\sum_{j=1}^n p_j^q} O_i. \quad (2)$$

Let the energy observable be  $\mathcal{U}$  and let it take on the value  $\varepsilon_i$  in the  $i$ -th state. Then maximization of the entropy under the constraints of fixed average energy,  $\langle \mathcal{U} \rangle = U$ , and of normalization,  $1 = \sum_{i=1}^n p_i$ , gives the  $q$ -exponential distribution,

$$p_j = \frac{(1 - (1-q) Z_q^{q-1} \beta (\varepsilon_j - U))_+^{\frac{1}{1-q}}}{Z_q}, \quad (3)$$

with

$$Z_q = \sum_{j=1}^N (1 - (1-q) Z_q^{q-1} \beta (\varepsilon_j - U))_+^{\frac{1}{1-q}}. \quad (4)$$

(Here, the notation  $(x)_+$  means  $x$  if  $x > 0$  and zero otherwise.) It is straightforward to show that if the energy levels are functions of some parameter,  $\varepsilon_i = \varepsilon_i(\lambda)$ , then

$$\frac{dU}{d\lambda} = \beta^{-1} \frac{dS}{d\lambda} + \frac{dW}{d\lambda}, \quad (5)$$

where the work is defined as  $dW = \sum_{i=1}^n p_i d\varepsilon_i$ . This is recognized as the first law of thermodynamics.

The peculiar nature of the average, based as it is on the escort distribution, naturally suggests a reformulation of the theory. If one defines the new quantity,

$$P_i = \frac{p_i^q}{\sum_{j=1}^n p_j^q}, \quad (6)$$

which is invertible,

$$p_i = \frac{P_i^{1/q}}{\sum_{j=1}^n P_j^{1/q}}, \quad (7)$$

then the Tsallis entropy becomes

$$\tilde{S}_q = \frac{1 - \left( \sum_{i=1}^n P_i^{1/q} \right)^{-q}}{q-1} \quad (8)$$

and averages are computed as normal,

$$\langle \mathcal{O} \rangle = \sum_{i=1}^n P_i O_i. \quad (9)$$

Extremizing this entropy subject to the usual constraints again gives a  $q$ -exponential distribution, but with exponent  $2 - 1/q$  rather than  $q$ . In the following, we refer to the original, more familiar formulation of non-extensive thermodynamics in terms of the Tsallis entropy as the “little- $p$ ” picture and the reformulation given here as the “big- $P$ ” picture. All of this has long been known in the literature of non-extensive thermodynamics [10]. The little- $p$  formulation is generally favored because of one important difference: the Tsallis entropy is a concave function of the probabilities whereas the same is not true of the big- $P$  entropy. (Concavity is assumed to be required of a generalized entropy even though the connection between concavity and thermodynamic stability is complicated by the non-additivity of the entropy [11].)

On the other hand, there is another important difference between the two pictures from a more physical point of view: the escort probabilities have the interpretation of being a measure of the likelihood of finding the system in a given microstate. This is obvious if one considers an ensemble of systems in which case the fact that the averages are computed via eq. (9) implies that the fraction of systems in microstate  $i$  must be  $P_i$ . This is to say that the usual ensemble interpretation of statistical averages implies a frequentist interpretation of the escort probabilities. Conversely no such ontology can be imposed on the small- $p$  probabilities. When using the small- $p$  formulation to express the average  $\langle \mathcal{O} \rangle = \sum_{i=1}^n \frac{p_i^q}{\sum_{j=1}^n p_j^q} O_i$ , it is clear that  $p_i$  does not appear to measure the frequency of anything. Furthermore, experiments, which measure averages, are always going to be determining the escort (big- $P$ ) distribution and not the small- $p$  distribution. The fact that both are  $q$ -exponentials means that this distinction is not of practical importance, but the distinction is real. The question then arises, is this distinction ever of practical importance? We next show that it plays a critical role in the discussion of stability of the entropy functional.

**Stability.** – The discussion of stability is based on ideas first introduced by Lesche while studying the Renyi entropy [12]. Lesche-stability is defined in terms of countable families of probability distributions,  $W_n = (w_1^{(n)}, \dots, w_n^{(n)})$ . An entropy function,  $S^{(n)}(W_n)$  is said to be Lesche-stable if given any two probability distributions,  $W_n$  and  $W'_n = (w_1^{(n)'}, \dots, w_n^{(n)'})$ , for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\delta > |W_n - W'_n| \equiv \sum_i \left| w_i^{(n)} - w_i^{(n)'} \right| \quad (10)$$

implies that

$$\epsilon > \left| \frac{S^{(n)}(W_n) - S^{(n)}(W'_n)}{S_{\max}^{(n)}} \right|, \quad (11)$$

where  $S_{\max}^{(n)}$  is the maximum value possible for the entropy functional. The quantity  $|W_n - W'_n|$  in eq. (10) is the

$L_1$  norm, and the implication (10)  $\rightarrow$  (11) is required to hold with fixed  $\epsilon, \delta$  for all  $n$  including the limit of  $n \rightarrow \infty$ . This definition is technical, but the idea is simple: when the the probabilities change, the change in the entropy should be small if the change in the probabilities is small. By formulating the definition in terms of a limit, it covers both the case of probability assignments with finite support (*i.e.* only some fixed number of elements are nonzero) as well as assignments with infinite support. Lesche proved that the Boltzmann entropy is stable, but gave a simple example showing that the Renyi entropy is not stable [12]. Abe has given a proof that the Tsallis entropy in the small- $p$  picture is stable [7].

One question that arises immediately is why use the  $L_1$  norm in measuring the distance between two distributions? In his original paper [12], Lesche stated that this measure can be related to the difficulty in experimentally distinguishing two probability assignments. The full argument was given recently by Abe, Lesche and Mund [13] and goes as follows. To tell whether  $W$  or  $W'$  is the correct description of a given ensemble of systems, one performs some number,  $Z$ , of experiments which, in each instance, determine which microstate the system is in. Of course, microstates are usually not directly observable, but for example, one might measure the energy and from this deduce the microstate (or a set of compatible microstates). The result is a sequence of observed microstates,  $S \equiv (s_1, \dots, s_Z)$ . If the correct distribution is  $W$ , the probability to observe this sequence is the product of the probabilities of observing each microstate,  $P(S; W) = w_{s_1} w_{s_2} \dots w_{s_Z}$ . Abe *et al.* define the test of whether or not the distribution is  $W$  by requiring that, prior to the experiment, the experimenter specify a collection of possible results,  $C = \{S^{(j)}\}$ . The probability that the observed  $S \in C$  is  $\alpha = \sum_j P(S_j; W)$ , while the probability  $\beta = 1 - \sum_j P(S_j; W')$  is the probability that, if the true distribution is  $W'$ , the observation will not be in the set  $C$ . In order to be an effective test, the set  $C$  must be constructed so that both  $\alpha$  and  $\beta$  are large: *i.e.*, so that if  $W$  is correct, the observation is likely to be in  $C$  and if  $W'$  is correct, the observation is likely to not be in  $C$ . They show that for given  $Z$ , no such test is possible if the  $L_1$  difference between  $W$  and  $W'$  is too small.

The key observation here is that the relevance of the  $L_1$  metric is based on an estimation of probabilities of observing certain outcomes, that is on the basis of a frequentist measure. Therefore, within the framework of non-extensive thermodynamics, these probabilities necessarily correspond to the escort probabilities. Thus, to test Lesche-stability in this formalism, the  $L_1$  metric must be applied to the escort probabilities. This is because the big- $P$  probabilities,  $P$ , are the *physical* probabilities measured in an experiment. Demonstrations based on separation of the small- $p$  distributions measured by the  $L_1$  metric do not correspond to the concept of Lesche-stability. An alternative statement, entirely within the little- $p$  picture, is that one must check stability of the

Tsallis entropy using the metric

$$|p - p'| = \sum_{i=1}^n \left| \frac{p_i^q}{\sum_{j=1}^N p_j^q} - \frac{p_i'^q}{\sum_{j=1}^N p_j'^q} \right|. \quad (12)$$

*The Tsallis entropy is unstable.* So, is the Tsallis entropy Lesche-stable? The answer is that it is not. Consider the case  $q < 1$  and let  $P$  be the uniform distribution,  $P_i = \frac{1}{n}$  for all  $i$ , and  $P'$  be given by  $P'_1 = \frac{1}{n} + \frac{\delta}{2}$  and  $P'_i = \frac{1}{n} - \frac{\delta}{2(n-1)}$  for  $n \geq i > 1$ . Then, it is obvious that

$$|P - P'|_{L_1} = \delta \quad (13)$$

and

$$\tilde{S}_q(P) = \frac{1 - n^{1-q}}{q-1}. \quad (14)$$

This happens to be equal to  $\tilde{S}_{q, \max}$ . The perturbed distribution gives

$$\tilde{S}_q(P') = \frac{1 - \left( \left( \frac{1}{n} + \frac{\delta}{2} \right)^{1/q} + (n-1) \left( \frac{1}{n} - \frac{\delta}{2(n-1)} \right)^{1/q} \right)^{-q}}{q-1} = \frac{1 - \left( \frac{\delta}{2} \right)^{-1} \left( 1 + \mathcal{O} \left( \frac{\delta^{-1}}{n^{1-q}} \right) \right)}{q-1}. \quad (15)$$

Hence,

$$\lim_{n \rightarrow \infty} \left| \frac{\tilde{S}_q(P) - \tilde{S}_q(P')}{\tilde{S}_{q, \max}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n^{1-q} - \left( \frac{\delta}{2} \right)^{-1}}{n^{1-q}} \right| = 1. \quad (16)$$

So, no matter how close the distributions (*i.e.* no matter how small  $\delta$ ), the difference between the entropies is finite. *There is no difference if the calculation is translated into the small- $p$  picture: the only substantive issue is whether the difference in the small- $p$  or the big- $P$  quantities is evaluated using the  $L_1$  metric.*

**The homogeneous entropy.** – To summarize, our arguments show that the Tsallis entropy with linear averages is Lesche-stable (as proven by Abe) but with the escort distribution it is Lesche-unstable. On the other hand, it is known that both formalisms give negative specific heats giving rise to questions of thermodynamic stability [2,4,11]. In fact, this was one issue that led to the search for an alternative to linear averages [3]. So, is there a non-extensive formalism that is both Lesche-stable and that gives positive specific heats? Given the intimate connection between the concept of Lesche-stability and the physical probabilities, one might wonder if it would make more sense to use the big- $P$  entropy, eq. (8) together with the linear averages as a starting point. As it happens, this is unsatisfactory because of the fact that the big- $P$  entropy functional is not concave [8]. There is, however, a closely related function known in the information theory literature as the homogeneous entropy [14] given by

$$S_q^H(P) = \frac{\left( \sum_{i=1}^n P_i^{1/q} \right)^q - 1}{q-1}, \quad (17)$$

which is concave for all positive  $q$ , is maximized by the uniform distribution and is extensible [15]. Our proposal is that this be used in conjunction with the linear averaging procedure as a basis for the formulation of non-extensive thermodynamics. When maximized under constraint of normalization and fixed internal energy, the result is a  $q$ -exponential,

$$P_j = Z_q^{-1} (1 - (1 - q) Z_q^{1-q} \beta (\varepsilon_j - U))^{1/q}, \quad (18)$$

with

$$\begin{aligned} Z_q &= \left( \sum_{i=1}^n P_i^{1/q} \right)^{-q} = \\ &= \sum_i \left( 1 - (1 - q) \frac{\beta (\varepsilon_j - U)}{Z_q^{q-1}} \right)^{1/q} = \\ &= \sum_{i=1}^n \left( 1 - (1 - q) \frac{\beta (\varepsilon_j - U)}{Z_q^{q-1}} \right)^{1/q}. \end{aligned} \quad (19)$$

The equality of the second and third lines follows from the constraint on the average energy. From this expression, it is straightforward to show that

$$\frac{\partial S_q^H}{\partial U} = \beta, \quad (20)$$

so that the Lagrange multiplier  $\beta$  corresponds to the inverse of the thermodynamic temperature. Furthermore, the specific heat is

$$C_V = \frac{\partial U}{\partial T} = q Z_q^{-\frac{(q-1)^2}{q}} \sum_i (\beta (\varepsilon_j - U))^2 P_i^{\frac{2q-1}{q}}, \quad (21)$$

which is positive definite. It is also easy to show that if the energies  $\varepsilon_i$  are a function of some external parameter,  $\lambda$ , then

$$\frac{\partial U}{\partial \lambda} = \beta^{-1} \frac{\partial S_q^H}{\partial \lambda} + \sum_{i=1}^n p_j \frac{\partial \varepsilon_j}{\partial \lambda}, \quad (22)$$

which confirms the second law of thermodynamics for this model.

Lesche-stability is also easy to show. Let  $\delta_i = P'_i - P_i$ ,  $\sum |P'_i - P_i| = \delta < 1$  and assume without loss of generality that  $S(P') > S(P)$  where  $S(P) \equiv (q-1)(S_q^H(P) + 1)$ . Then, for  $q < 1$ , one has that

$$\begin{aligned} S(P') &= \left( \sum_{i=1}^n (P_i + \delta_i)^{1/q} \right)^q = \\ &= \left( \sum_{i=1}^n |P_i + \delta_i|^{1/q} \right)^q \leq \\ &= \left( \sum_{i=1}^n P_i^{1/q} \right)^q + \left( \sum_{i=1}^n |\delta_i|^{1/q} \right)^q. \end{aligned} \quad (23)$$

The last line follows from the Minkowski inequality (a generalization of the triangle inequality) [16]. For  $q < 1$  and  $|\delta_i| \leq \delta < 1$ , one has  $|\delta_i|^{1/q} < |\delta_i|$  so

$$S(P') \leq \left( \sum_{i=1}^n P_i^{1/q} \right)^q + \left( \sum_{i=1}^n |\delta_i| \right)^q = S(P) + \delta^q. \quad (24)$$

Lesche-stability follows immediately. For  $q > 1$  the proof is slightly more complicated. Note that if  $x \geq y > 0$  and  $q > 1$ , then  $q(x-y)x^{q-1} \geq x^q - y^q$ , which follows from the fact that  $q(x-y)x^{q-1} - x^q + y^q$  is monotonically decreasing as a function of  $y$ . Making all the same assumptions as above, this implies that

$$\begin{aligned} &\left( \sum_{i=1}^n P_i'^{1/q} \right)^q - \left( \sum_{i=1}^n P_i^{1/q} \right)^q \leq \\ &= q \left( \sum_{i=1}^n (P_i'^{1/q} - P_i^{1/q}) \right) \left( \sum_{i=1}^n P_i'^{1/q} \right)^{q-1}. \end{aligned} \quad (25)$$

Next, note that for  $q > 1$  and  $x > y > 0$ ,  $x^{1/q} - y^{1/q} \leq (x-y)^{1/q}$  as follows from the fact that  $(x-y)^{1/q} - x^{1/q} + y^{1/q}$  is convex as a function of  $y$ . Thus

$$\begin{aligned} &\left( \sum_{i=1}^n P_i'^{1/q} \right)^q - \left( \sum_{i=1}^n P_i^{1/q} \right)^q \leq \\ &= q \left( \sum_{i=1}^n |\delta_i|^{1/q} \right) \left( \sum_{i=1}^n P_i'^{1/q} \right)^{q-1}. \end{aligned} \quad (26)$$

The two sums on the right can be bounded by maximizing  $\sum_{i=1}^n x_i^{1/q}$  subject to  $\sum_{i=1}^n x_i = \gamma$  using a Lagrange multiplier. The result is that  $x_i = \gamma/n$  and the sum is  $\gamma^{1/q} n^{(q-1)/q}$  giving

$$\begin{aligned} &\left( \sum_{i=1}^n P_i'^{1/q} \right)^q - \left( \sum_{i=1}^n P_i^{1/q} \right)^q \leq \\ &= q \delta^{1/q} \left( n^{(q-1)/q} \right)^q = q \delta^{1/q} n^{q-1}, \end{aligned} \quad (27)$$

which, after normalization, implies Lesche-stability.

Finally, we can make contact with the Tsallis entropy as follows. The normalization condition can be enforced by eliminating one of the degrees of freedom, *i.e.* by using

$$P_n = 1 - \sum_{i=1}^{n-1} P_i. \quad (28)$$

However, this is awkward as it treats one degree of freedom differently from the others. A more symmetrical way to impose it would be to introduce auxiliary quantities,  $u_i$ , and to write

$$P_i = \frac{u_i}{\sum_{j=1}^n u_j}. \quad (29)$$

This is completely general. Note that it is degenerate as  $\{u_i\}$  and  $\{\lambda u_i\}$  give the same  $\{P_i\}$ . Since  $P_i > 0$ , all of the

$u_i$  must be either positive or negative. Again, without loss of generality, we can take them all to be positive. In terms of these, the entropy becomes

$$S_q^H(p) = \frac{\left(\sum_{i=1}^n u_i^{1/q}\right)^q \left(\sum_{j=1}^n u_j\right)^{-1} - 1}{q-1}. \quad (30)$$

We can simplify this by using the freedom mentioned above to impose the single constraint,

$$1 = \sum_{i=1}^n u_i^{1/q}, \quad (31)$$

which is equivalent to choosing a particular scaling of the  $u$ 's. Note that since all the  $u_i > 0$ , this implies that they are all also less than one. Then, it appears to be more convenient to introduce  $v_i = u_i^{1/q} \in [0, 1]$  giving

$$P_i = \frac{v_i^q}{\sum_{j=1}^n v_j^q},$$

$$S_q^H(v) = \frac{\left(\sum_{j=1}^n v_j^q\right)^{-1} - 1}{q-1} = \frac{1 - \sum_{j=1}^n v_j^q}{(q-1)\sum_{j=1}^n v_j^q}, \quad (32)$$

$$1 = \sum_{i=1}^n v_i,$$

$$\langle \mathcal{O} \rangle = \sum_{i=1}^n \frac{v_i^q}{\sum_{j=1}^n v_j^q} O_i.$$

The ‘‘escort probabilities’’ therefore arise naturally as a way of encoding the normalization constraint. However, it is clear in this interpretation that the  $\{v_i\}$  are not physical probabilities but just quantities that happen to be positive and to sum to one. The form of the homogeneous entropy written in terms of the  $\{v_i\}$  is known in the literature and is called the ‘‘normalized Tsallis entropy’’ [17]. However, note that the normalized Tsallis entropy is only concave for  $q \leq 1$  so the homogeneous entropy is a more general starting point.

**Concluding comments.** – Our conclusion is that when the concept of Lesche stability is properly applied within the usual formalism of non-extensive thermodynamics, the Tsallis entropy is just as unstable as the Renyi entropy originally considered by Lesche. Whether or not this conclusion, based on particular admittedly artificial examples, is physically relevant and would have practical implications is a question of on-going debate [18] just as was the question of the stability of the Renyi entropy [19]. However, arguments such as those given in [18] should be reconsidered in light of the correct application of the  $L_1$  measure.

A further consequence which follows from the big- $P$  formulation concerns the stability of the averages of observables. In a recent demonstration they were shown

to be unstable in the non-extensive formalism [6]. This result was obtained using the small- $p$  picture where the interpretation of the probabilities is problematic, while it was shown that in the classical formulation of statistical mechanics with linear averages, stability was guaranteed. This is precisely in accordance with our arguments: since the  $L_1$  measure should be applied to the escort probabilities (*i.e.* the big  $P$ 's), it follows that the averages are stable.

In summary, the use of the  $L_1$  norm in framing the concept of Lesche-stability is justified by considering an experimental test designed to distinguish different hypothesized probability distributions. As such, it is a property not only of the form of the entropy but also of the means used to relate the ‘‘probabilities’’ occurring in the entropy to experiment. Our conclusion is that the result of Abe [7] should be interpreted as demonstrating that the combination of Tsallis entropy and linear averages is stable while the combination of Tsallis entropy with the escort distribution averages is unstable (as shown by our example above). This being the case, the further observations concerning the stability of the averages [6] only reinforces these conclusions.

On the other hand, we have shown that the homogeneous entropy with the usual linear averaging procedure provides a satisfactory starting point for the development of non-extensive thermodynamics. We note that this is not the first time the homogeneous entropy has occurred in the context of non-extensive thermodynamics. For example, Lavenda and Dunning-Davies have used it as an example illustrating that certain features of the Tsallis entropy are not unique [20]. In the form of the normalized Tsallis distribution, it appears to have first been discussed by Rajagopal and Abe [17] where it was noted that it is only concave for  $q \in [0, 1]$ . Abe subsequently concluded that it is not Lesche-stable, but by our interpretation, this argument shows instability when the linear averaging procedure is used [7]. We also note that Lenzi *et al.* had already demonstrated that the normalized Tsallis entropy gives positive specific heats [21]. Our main contribution has been to note that the homogeneous entropy is more general than the normalized Tsallis entropy (because it is concave for all values of  $q$ ) and to show it is Lesche-stable. Together with other properties, such as the positivity of the specific heats, we suggest this makes it a preferred starting point for the development of non-extensive thermodynamics.

Finally, this approach also sheds light on the underlying ontology of the formalism. The concept of the escort distribution has become an accepted part of the non-extensive formalism but, as we have discussed, it is hard to understand in what sense the small- $p$  quantities are ‘‘probabilities’’ as opposed to a set of quantities that happen to be positive and normalized. Nevertheless, various arguments have also been given for using the escort distribution rather than the linear averaging procedure [22,23]. In our formulation, this ambiguity is clarified: the only *probabilities* are the physical probabilities

used to compute averages. These are what we have called the big- $P$  probabilities,  $P$ , and they are what is measured in experiment. The escort distribution arises as a natural way to simplify the form of the entropy, but the equivalent of the small- $p$  variables are clearly quantities to which no physical significance attaches.

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