Transport properties of dense dissipative hard-sphere fluids for arbitrary energy loss models

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The revised Enskog approximation for a fluid of hard spheres which lose energy upon collision is discussed for the case that the energy is lost from the normal component of the velocity at collision but is otherwise arbitrary. Granular fluids with a velocity-dependent coefficient of restitution are an important special case covered by this model. A normal solution to the Enskog equation is developed using the Chapman-Enskog expansion. The lowest order solution describes the general homogeneous cooling state and a generating function formalism is introduced for the determination of the distribution function. The first order solution, evaluated in the lowest Sonine approximation, provides estimates for the transport coefficients for the Navier-Stokes hydrodynamic description. All calculations are performed in an arbitrary number of dimensions.

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I. INTRODUCTION

Hydrodynamics is important because it provides a connection between microscopic models of particle interactions and experimentally observable behavior. Its experimental significance is due to the fact that hydrodynamics describes the evolution of fundamental quantities, including local mass, momentum and energy density which are important in a variety of applications from microfluids to cosmology. At a microscopic level, these quantities are "slow" variables which evolve on a time scale which is well separated from faster, microscopic variables so that the effect of the latter can be adequately encapsulated in the various transport coefficients appearing in the hydrodynamic description. Recently, much interest has centered on the hydrodynamics of granular media which are characterized by a loss of energy during collisions between the grains [1–3]. The simplest microscopic model of granular materials consists of hard-sphere grains which lose a fixed fraction ε of the part of the kinetic energy associated with the longitudinal, or normal, component of the velocities at the moment of collision while the transverse components of the velocities of the colliding particles are unchanged. The transport properties of this model system, which will be referred to as a "simple granular gas" below, have been worked out in some detail for the D-dimensional low-density fluid [4,5], the dense single-component [6] and low-density binary [7] fluids. However, there is interest in more complex models in which the fractional energy loss ε is itself a function of the normal kinetic energy as such models are apparently more realistic [8–10]. Furthermore, there are a wealth of phenomena, such as endothermic and exothermic chemical reactions, in which kinetic energy gets converted to some other form and thus couples, e.g., chemical reactions and hydrodynamics. A good example is sonoluminescence [11,12] where classical hydrodynamics is often used, with some success [13], to try to understand the complex processes taking place under extreme conditions. Recently, the possible scattering laws for dissipative collisions consistent with conservation of momentum and angular momentum has been discussed and formally exact expressions for the balance equations of mass, momentum, energy and species have been formulated and the Enskog approximation discussed [14]. The purpose of the present work is to show that the Chapman-Enskog expansion [15] can be applied to that kinetic theory for the case of a one-component fluid in D dimensions with an arbitrary model for the normal energy loss and reduced to a relatively simple form thereby providing convenient expressions for the transport coefficients covering this entire class of models. These results will therefore include as special cases the known results for elastic hard spheres in two and three dimensions [16,15] and for the simple granular gas in three dimensions [6] as well as such interesting models as those with velocity-dependent coefficient of restitution [8–10] for which the Enskog equation has not previously been solved.

The emphasis on instantaneous interactions is due to their unique properties. Instantaneous hard-core interactions can be described in the Enskog approximation which is a finite density approximation known to give reasonable results for moderately dense fluids [15] and which can even describe hard-core solids [17]. For most other interactions, only the Boltzmann description, a low density approximation, is available. The fact that the Enskog kinetic model is applicable to solids means that there is scope for application to dense granular systems, which might involve jamming, as well as to extreme conditions such as occur during sonoluminescence. Thus, despite its artificial nature, the hard-core model is an important tool in understanding the real world.

Although the work presented below is intended to be applicable to a variety of phyically interesting systems, it is certainly true that the most important example of nonconservative, hard-core collision models is as a model of granular fluids. In this case, there has long been a debate as to the applicability of Enskog-type kinetic theory and the hydrodynamics that results from it (see, e.g., Ref. [18]). Support for these concerns comes from studies, e.g., the work of Soto and Mareschal [19] and Pagonabarraga *et al.* [20], showing the existence of velocity correlations of a type explicitly ig-

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nored in the derivation of the Enskog equation [21]. However, this must be balanced by the fact that these correlations are not particular to granular systems, but are a generic feature of non-equilibrium fluids in general [22] and in all cases (granular fluids, sheared elastic fluids, etc.) are expected to increase with importance with increasing density and increasing distance from equilibrium. Nevertheless, the success of Enskog-level kinetic theory, and of the hydrodynamics derived from it, in describing the properties of low and moderately dense nonequilibrium fluids has long been recognized. For example, good agreement has been demonstrated between the predictions of Enskog theory and the results of computer simulation for the shear-rate dependent shear viscosity and viscoelastic functions of both sheared elastic fluids [23,24] and granular fluids (including polydisperse granular fluids) [25]. Similar comparisons of time-dependent phenomena also support this conclusion [26] and the technical question of the existence of hydrodynamics modes has been addressed for low-density systems [27]. Further evidence for the applicability of the Enskog description comes from the recent experimental investigations of Yang *et al.* [28] and Huan *et al.* [29] which show that observed profiles of density and temperature in vibrated granular systems are well described by hydrodynamics derived from Enskog theory. All of this, together with older studies of the same questions concerning nonequilibrium elastic hard-sphere systems, see, e.g., Refs. [30,31], lend support to they belief that Enskog-level kinetic theory and the hydrodynamics derived from it provide a useful description of fluidized nonconservative hard-core systems up to moderate densities.

In Sec. II of this paper, the possible scattering laws, exact balance laws and consequent Enskog equations are reviewed. The general form of the Chapman-Enskog expansion is also discussed. The zeroth-order Chapman-Enskog result is just the exact description of a spatially homogeneous system and is discussed in Sec. III. For an equilibrium system, this is the Maxwell distribution but when there is energy loss, the fluid cools and the resulting homogeneous but nonstationary state is known as the homogeneous cooling state (HCS). Section III formulates the description of the HCS in terms of a generating function formalism and gives all information needed to calculate the simplest corrections to the Maxwell distribution for arbitrary energy-loss models. In Sec. IV, the Chapman-Enskog expansion is extended to first order which is sufficient to get the Navier-Stokes transport properties. The general formalism is illustrated by application to the simple granular gas in D dimensions. The paper concludes with a discussion of the approximations made and comparison to more complete calculations.

II. THEORETICAL BACKGROUND

A. Kinetic theory with energy loss

We consider a collection of particles having mass m and hard sphere diameter σ which interact via instantaneous collisions. The position and velocity of the ith particle will be denoted by $\vec{q}_i$ and $\vec{v}_i$, respectively, and the combined phase variable $(\vec{q}_i,\vec{v}_i)$ will sometimes be denoted as x_i. The only scattering law allowing for energy loss that is still consistent with conservation of total momentum and angular momentum [14] is that two particles having relative velocity $\vec{v}_{12}=\vec{v}_1-\vec{v}_2$ prior to collision must have relative velocity

$$\vec{v}_{12}'=\vec{v}_{12}-\hat{q}_{12}\left(\vec{v}_{12}\cdot\hat{q}_{12}+\mathrm{sgn}(\vec{v}_{12}\cdot\hat{q}_{12})\sqrt{(\vec{v}_{12}\cdot\hat{q}_{12})^2-\frac{4}{m}\delta E}\right) \tag{1}$$

after the collision, where $\hat{q}_{12}$ is a unit vector pointing from the center of the first atom to the center of the second atom. The energy loss is δE which we allow in general to be a function of the normal relative velocity

$$\delta E=\Delta(\vec{v}_{12}\cdot\hat{q}) \tag{2}$$

and the center of mass velocity $\vec{V}_{12}=\vec{v}_1+\vec{v}_2$ is unchanged. It is easy to confirm that the change of energy upon collision is

$$\tfrac{1}{2}mv_1'^2+\tfrac{1}{2}mv_2'^2=\tfrac{1}{2}mv_1^2+\tfrac{1}{2}mv_2^2-\delta E. \tag{3}$$

The simple granular fluid model is based on the energy loss function

$$\Delta(x)=(1-\alpha^2)\frac{m}{4}x^2, \tag{4}$$

where the constant α is the coefficient of restitution. More complex models involve a velocity dependent α while other choices of energy loss function, involving, e.g., a thresholds for energy loss, would be appropriate for chemical reactions. In the following, it is convenient to introduce the momentum transfer operator $\hat{b}$ defined for any function of the velocities $g(\vec{v}_1,\vec{v}_2;t)$ by

$$\hat{b}g(\vec{v}_1,\vec{v}_2;t)=g(\vec{v}_1',\vec{v}_2';t). \tag{5}$$

In some applications, the energy loss may not occur for all collisions but rather might be a random occurrence. We will therefore also consider throughout a somewhat generalized model in which for any particular collision, the energy loss function is randomly chosen from a set of possible functions $\{\Delta_a(x)\}$ with probability $K_a(\hat{q}_{12}\cdot\vec{v}_{12})$ which may itself, as indicated by the notation, depend on the dynamic variable $\hat{q}_{12}\cdot\vec{v}_{12}$. A simple case would be that a fixed fraction $K_1=1-p$ of collisions are elastic with energy loss function $\Delta_1(x)=0$ while the remainder occur with probability $K_2=p$ are inelastic with energy loss $\Delta_2(x)\neq 0$. In any case, it is assumed that $\Sigma_a K_a(x)=1$ for all x. The momentum transfer operator for the type a collisions will be written as $\hat{b}_a$.

The kinetic theory, Liouville equation and the Enskog approximation, for arbitrary energy loss function has been discussed in Ref. [14]. The one-body distribution function $f(\vec{q}_1,\vec{v}_1;t)$, giving the probability to find a particle at position $\vec{q}_1$ with velocity $\vec{v}_1$ at time t, satisfies, in the Enskog approximation, the kinetic equation

$$\left(\frac{\partial}{\partial t}+\vec{v}_1\cdot\frac{\partial}{\partial\vec{q}_1}\right)f(x_1;t)=J[f,f], \tag{6}$$

where the shorthand notation $x_1=(\vec{q}_1,\vec{v}_1)$ is used. The collision operator is

$$J[f,f] = -\int dx_2 \bar{T}_-(12)\chi(\vec{q}_1,\vec{q}_2;[n])f(\vec{q}_1,\vec{v}_1;t)f(\vec{q}_2,\vec{v}_2;t),$$

where $\chi(\vec{q}_1,\vec{q}_2;[n])$ is the local equilibrium pair distribution function which is in general a functional of the local density. The binary collision operator $\bar{T}_-(12)$ is

$$\bar{T}_-(12) = \left(\sum_a J_a(\vec{v}_1,\vec{v}_2)(\hat{b}_a)^{-1}K_a(\hat{q}_{12}\cdot\vec{v}_{12}) - 1\right) \times\Theta(-\vec{v}_{12}\cdot\vec{q}_{12})\delta(q_{12}-\sigma)\vec{v}_{12}\cdot\hat{q}_{12}, \tag{7}$$

where $\Theta(x)$ is the step function, $(\hat{b}_a)^{-1}$ is the inverse of the momentum exchange operator and J_a is the Jacobian of the inverse collision

$$J_a(\vec{v}_1,\vec{v}_2) = \left|\frac{\partial(\hat{b}_a\vec{v}_1,\hat{b}_a\vec{v}_2)}{\partial(\vec{v}_1,\vec{v}_2)}\right|^{-1}. \tag{8}$$

It will not be necessary to work much with this complicated operator as most calculations can make use of its simpler adjoint $T_+(12)$ defined for arbitrary functions of the phase variables $A(x_1,x_2)$ and $B(x_1,x_2)$ by

$$\int dx_1dx_2A(x_1,x_2)\bar{T}_-(12)B(x_1,x_2) = -\int dx_1dx_2B(x_1,x_2)T_+(12)A(x_1,x_2) \tag{9}$$

so that

$$T_+(12) = \Theta(-\vec{v}_{12}\cdot\vec{q}_{12})\delta(q_{12}-\sigma)(-\vec{v}_{12}\cdot\hat{q}_{12}) \times\left(\sum_a K_a(\hat{q}_{12}\cdot\vec{v}_{12})\hat{b}_a - 1\right). \tag{10}$$

B. Hydrodynamic fields and balance equations

The hydrodynamic fields are the local mass density $\rho(\vec{r},t)$, the local velocity field $\vec{u}(\vec{r},t)$, and the local temperature field $T(\vec{r},t)$. They are defined in terms of the distribution by

$$\rho(\vec{r},t) = mn(\vec{r},t) = m\int d\vec{v}_1 f(\vec{r},\vec{v}_1;t),$$

$$\rho(\vec{r},t)\vec{u}(\vec{r},t) = m\int d\vec{v}_1\vec{v}_1 f(\vec{r},\vec{v}_1;t),$$

$$\frac{D}{2}n(\vec{r},t)k_BT(\vec{r},t) = \frac{1}{2}m\int d\vec{v}_1V_1^2 f(\vec{r},\vec{v}_1;t), \tag{11}$$

where $n(\vec{r},t)$ is the local number density and D is the number of dimensions. In the third equation $\vec{V}_1=\vec{v}_1-\vec{u}(\vec{r},t)$. Their time evolution follows from that of the distribution and is given by [14]

$$\frac{\partial}{\partial t}n + \vec{\nabla}\cdot(\vec{u}n) = 0,$$

$$\frac{\partial}{\partial t}\rho\vec{u} + \vec{\nabla}\cdot(\rho\vec{u}\vec{u}) + \vec{\nabla}\cdot\overleftrightarrow{P} = 0,$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)T + \frac{2}{Dnk_B}[\overleftrightarrow{P}:\vec{\nabla}\vec{u} + \vec{\nabla}\cdot\vec{q}] = \xi. \tag{12}$$

The pressure tensor is the sum of two contributions $\overleftrightarrow{P}=\overleftrightarrow{P}^K+\overleftrightarrow{P}^C$ with the kinetic part being

$$\overleftrightarrow{P}^K(\vec{r},t) = m\int d\vec{v}_1 f_l(\vec{r},\vec{v}_1,t)\vec{V}_1\vec{V}_1, \tag{13}$$

and the collisional contribution being

$$\overleftrightarrow{P}^C(\vec{r},t) = -\frac{m}{4V}\sigma\sum_a\int dx_1dx_2\hat{q}_{12}\hat{q}_{12}(\hat{q}_{12}\cdot\vec{v}_{12})\delta(q_{12}-\sigma)\Theta(-\hat{q}_{12}\vec{v}_{12})\chi(\vec{q}_1,\vec{q}_2;[n])f(x_1;t)f(x_2;t)K_a(\hat{q}_{12}\cdot\vec{v}_{12}) \times\int_0^1 dy\,\delta(\vec{r}-y\vec{q}_1-(1-y)\vec{q}_2)\left(-\vec{v}_{12}\cdot\hat{q}_{12} - \text{sgn}(\vec{v}_{12}\cdot\hat{q}_{12})\sqrt{(\vec{v}_{12}\cdot\hat{q}_{12})^2 - \frac{4}{m}\Delta_a(\hat{q}_{12}\cdot\vec{v}_{12})}\right). \tag{14}$$

Similarly, the heat flux has a kinetic contribution

$$\vec{q}^K(\vec{r},t) = \tfrac{1}{2}m\int d\vec{v}_1 f(\vec{r},\vec{v}_1,t)\vec{V}_1V_1^2 \tag{15}$$

and a collisional contribution

$$\vec{q}^C(\vec{r},t) = -\frac{m}{4V}\sigma\sum_a \int dx_1 dx_2 \hat{q}_{12}(\hat{q}_{12}\cdot\vec{v}_{12})\delta(q_{12}-\sigma)\Theta(-\hat{q}_{12}\cdot\vec{v}_{12})\chi(\vec{q}_1,\vec{q}_2;[n])f(x_1;t)f(x_2;t)K_a(\hat{q}_{12}\cdot\vec{v}_{12})$$
$$\times\int_0^1 dx\,\delta(\vec{r}-x\vec{q}_1-(1-x)\vec{q}_2)\frac{1}{2}(\vec{V}_1+\vec{V}_2)\cdot\hat{q}_{12}\left(-\vec{v}_{12}\cdot\hat{q}_{12}-\mathrm{sgn}(\vec{v}_{12}\cdot\hat{q}_{12})\sqrt{(\vec{v}_{12}\cdot\hat{q}_{12})^2-\frac{4}{m}\Delta_a(\hat{q}_{12}\cdot\vec{v}_{12})}\right). \quad (16)$$

Finally, because of the possibility of energy loss, the equation for the temperature includes a source term given by

$$\xi(\vec{r},t) = \frac{1}{2V}\sum_a \int dx_1 dx_2(\hat{q}_{12}\cdot\vec{v}_{12})\delta(q_{12}-\sigma)\Theta(-\hat{q}_{12}\cdot\vec{v}_{12})$$
$$\times K_a(\hat{q}_{12}\cdot\vec{v}_{12})\Delta_a(\hat{q}_{12}\cdot\vec{v}_{12})\chi(\vec{q}_1,\vec{q}_2;[n])$$
$$\times f(x_1;t)f(x_2;t)K_a(x_{12})\delta(\vec{r}-\vec{q}_1). \quad (17)$$

All of these expressions are exact, given the Enskog approximation, and show that the hydrodynamics of the system is completely specified once the one-body distribution is known.

C. Chapman-Enskog expansion

The Chapman-Enskog expansion is basically a gradient expansion of the kinetic equation assuming a particular form for the solution. Specifically, one attempts to construct a so-called normal solution in which all space and time dependence occurs through the hydrodynamic fields

$$f(\vec{r},\vec{v};t) = f(\vec{v}|\vec{r},\psi_t), \quad (18)$$

where the compact notation for the set of hydrodynamic fields $\psi_t(\vec{r})=\{n(\vec{r},t),T(\vec{r},t),\vec{u}(\vec{r},t)\}$ has been introduced and the notation indicates that the distribution is a functional of the hydrodynamic fields at time t. This means that time derivatives will be evaluated as

$$\frac{\partial}{\partial t}f(\vec{r},\vec{v};t) = \sum_i \int d\vec{r}'\frac{\partial\psi_{t,i}(\vec{r}')}{\partial t}\frac{\delta}{\delta\psi_{t,i}(\vec{r}')}f(\vec{r},\vec{v};t). \quad (19)$$

To order the terms in the gradient expansion, we introduce a uniformity parameter ϵ and replace $\vec{\nabla}$ with $\epsilon\vec{\nabla}$ and order terms in ϵ. Since the space and time derivatives are related by the balance equations, we also introduce an expansion of the time derivative $\partial/\partial t=\partial_t^{(0)}+\epsilon\partial_t^{(1)}+\cdots$ as well as of the distribution itself

$$f(\vec{q}_1,\vec{v}_1,t) = f_0[\vec{v}_1|\vec{q}_1,\psi_t] + \epsilon f_1[\vec{v}_1|\vec{q}_1,\psi_t] + \cdots . \quad (20)$$

Finally, in the Enskog approximation the collision operator is nonlocal and so must also be expanded (see Appendix A) as $J[f,f]=J_0[f,f]+\epsilon J_1[f,f]+\cdots$. Substituting these expansions into the Enskog equation and equating terms order by order in ϵ gives a set of equations for the distribution, the first two of which are

$$\partial_t^{(0)}f_0(x_1;t) = J_0[f_0,f_0],$$

$$\left(\partial_t^{(1)}+\vec{v}_1\cdot\frac{\partial}{\partial\vec{q}_1}\right)f_0(x_1;t)+\partial_t^{(0)}f_1(x_1;t)$$
$$= J_0[f_0,f_1]+J_0[f_1,f_0]+J_1[f_0,f_0]. \quad (21)$$

Since all time and space dependence of the distribution occurs via the hydrodynamic fields, the balance equations must also be expanded giving at zeroth order

$$\partial_t^{(0)}n = 0,$$

$$\partial_t^{(0)}n\vec{u} = 0,$$

$$\partial_t^{(0)}T = \frac{2}{Dnk_B}\xi^{(0)} \quad (22)$$

and at first order

$$\partial_t^{(1)}\rho + \vec{\nabla}\cdot(\vec{u}\rho) = 0,$$

$$\partial_t^{(1)}\rho\vec{u} + \vec{\nabla}\cdot(\rho\vec{u}\vec{u}) + \vec{\nabla}\cdot\overleftrightarrow{P}^{(0)} = 0,$$

$$(\partial_t^{(1)}+\vec{u}\cdot\vec{\nabla})T + \frac{2}{Dnk_B}[\overleftrightarrow{P}^{(0)}:\vec{\nabla}\vec{u}+\vec{\nabla}\cdot\vec{q}^{(0)}] = \frac{2}{Dnk_B}\xi^{(1)} \quad (23)$$

where, as noted, the fluxes and sources must also be expanded accordingly (see Appendix B). The logic of the normal solution is that these balance equations define the meaning of the time derivatives so that the time derivatives in Eq. (21) are evaluated using Eq. (23) and Eq. (19). Together with the expressions for the fluxes, Eqs. (13)–(17) suitably expanded, this gives a closed set of integrodifferential equations for the distribution function.

III. CHAPMAN-ENSKOG AT ZEROTH ORDER: THE HOMOGENEOUS COOLING STATE

A. Expansion of the zeroth-order distribution

At zeroth order in the gradient expansion, the distribution $f_0(x_1;t)$ must be a local function of the hydrodynamic fields so Eqs. (21) and (22) give

$$\left(\frac{2}{Dnk_B}\xi^{(0)}\right)\frac{\partial}{\partial T}f_0(x_1;t) = J_0[f_0,f_0] \quad (24)$$

with

$$\xi^{(0)}(\vec{r},t) = \frac{1}{2}\sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12}) \times K_a(\sigma\hat{q}\cdot\vec{v}_{12})\Delta_a(\sigma\vec{q},\vec{v}_{12}) \times \chi_0(\sigma;n(\vec{r}))f_0(\vec{r},\vec{v}_1;t)f_0(\vec{r},\vec{v}_2;t), \quad (25)$$

where $\chi_0(\sigma;n)$ is the pair distribution function for a uniform equilibrium fluid of density n. Notice that this is not only the zeroth order component of the Chapman-Enskog expansion, but that it is also an exact (within the Enskog approximation) equation for the distribution of a spatially homogeneous system. If there is no energy loss, the solution will simply be the Maxwell distribution. When energy is lost in collisions, and in the absence of external forcings, the system cools and this is commonly known as the homogeneous cooling solution (HCS).

To solve for the HCS, we expand the velocity dependence about an equilibrium distribution by writing it as

$$f_0(x_1) = f_M(v_1;\psi_t)\sum_i c_i(\psi_t)S_i\left(\frac{m}{2k_BT}v_1^2\right), \quad (26)$$

where the Maxwellian distribution

$$f_M(v_1;\psi_t) = n\pi^{-D/2}\left(\frac{2k_BT}{m}\right)^{-D/2}\exp\left(-\frac{m}{2k_BT}v_1^2\right)$$

and it is important to note that this depends on the exact local fields $\psi_t(\vec{r})$. The functions $\{S_i(x)\}_{i=0}^{\infty}$ comprise a complete set of polynomials which are orthogonal under a Gaussian measure so that

$$\int d\vec{v}f_M(v_1;\psi_t)S_i\left(\frac{m}{2k_BT}v_1^2\right)S_i\left(\frac{m}{2k_BT}v_1^2\right) = A_i\delta_{ij}, \quad (27)$$

where A_i is a normalization constant. In fact, these can be written in terms of the Sonine, or associated Laguerre, polynomials

$$L_k^{\alpha}(x) = \sum_{m=0}^{k}\frac{\Gamma(\alpha+k+1)(-x)^m}{\Gamma(\alpha+m+1)(k-m)!m!} \quad (28)$$

which satisfy

$$\int_0^{\infty} dx x^{\alpha}L_k(x)L_m^{\alpha}(x)\exp(-x) = \frac{\Gamma(\alpha+k+1)}{\Gamma(k+1)}\delta_{mk}, \quad (29)$$

so that in D dimensions

$$S_k(x) = L_k^{(D-2)/2}(x) \quad (30)$$

and

$$A_k = \frac{\Gamma\left(\frac{1}{2}D+k\right)}{\Gamma\left(\frac{1}{2}D\right)\Gamma(k+1)}. \quad (31)$$

Substituting Eq. (26) into the differential equation, Eq. (24), multiplying through by $L_k^{(D-2)/2}[(m/2k_BT)v_1^2]$ and integrating gives

$$\left(\frac{2}{Dnk_BT}\xi^{(0)}(\psi_t)\right)\left(T\frac{\partial}{\partial T}c_k(\psi_t) + k[c_k(\psi_t) - c_{k-1}(\psi_t)]\right) = \sum_{rs} I_{k,rs}(\psi_t)c_r(\psi_t)c_s(\psi_t) \quad (32)$$

with

$$I_{k,rs}(\psi_t) = -n^{-1}A_k^{-1}\int d\vec{v}_1 d2 L_k^{(D-2)/2}\left(\frac{m}{2k_BT}v_1^2\right)\overline{T}_-(12) \times \left(\frac{2k_BT}{m}\right)^{-D} f_M(v_1;\psi_t)f_M(v_2;\psi_t) \times L_r^{(D-2)/2}\left(\frac{m}{2k_BT}v_1^2\right)L_s^{(D-2)/2}\left(\frac{m}{2k_BT}v_1^2\right). \quad (33)$$

Notice that since $\xi^{(0)}\sim n^2$ and $I_{k,rs}\sim n$, the coefficients c_k can only depend on temperature so $c_k(\psi_t)=c_k[T(t)]$. Furthermore, it must be the case that $c_0=1$ and $c_1=0$ in order to satisfy the definitions of the hydrodynamic fields. It is easy to show that $I_{rs}^0=0$ so that the $k=0$ equation is trivial. Suppressing the dependence on ψ_t, the $k=1$ equation gives

$$-\frac{2}{Dnk_B}\xi^{(0)} = \sum_{rs} I_{1,rs}c_r c_s \quad (34)$$

and it may be confirmed that this is consistent with Eq. (25). The first nontrivial approximation is to take $c_k=0$ for $k>2$ and to use the $k=2$ equation to get

$$-\frac{2}{Dnk_BT}\xi^{(0)} = I_{1,00} + (I_{1,20}+I_{1,02})c_2,$$

$$\left(\frac{2}{Dnk_BT}\xi^{(0)}\right)\left(T\frac{\partial}{\partial T}c_2 + 2c_2\right) = I_{2,00} + (I_{2,20}+I_{2,02})c_2, \quad (35)$$

where terms involving c_2^2 on both sides of the equation are typically neglected as they are of similar structure to the neglected c_4 terms. For a simple granular fluid, there is no quantity with the units of energy except for the temperature, so the coefficients of the expansion, which are dimensionless, are temperature independent. For systems with additional energy scales, the coefficients must be determined by solving the resultant ordinary differential equations with appropriate boundary conditions. For example, if the energy loss were bounded, then at high temperatures it should be insignificant and one would expect $\lim_{T\to\infty}c_k(T)=\delta_{k0}$ to be the boundary condition.

B. Generating function formalism

The calculation of the integrals which define the coefficients on the right-hand side in Eq. (32) can be formulated in terms of a generating function. Specifically, it is shown in Appendix C that

$$I_{k,rs}(\psi_t) = -n^* \frac{\Gamma\left(\frac{1}{2}D\right)\Gamma(k+1)}{\Gamma\left(\frac{1}{2}D+k\right)}\left(\frac{2k_BT}{m\sigma^2}\right)^{1/2} \frac{1}{r!s!k!} \lim_{z_1\to 0}\lim_{z_2\to 0}\lim_{x\to 0} \frac{\partial^r}{\partial z_1^r}\frac{\partial^s}{\partial z_2^s}\frac{\partial^k}{\partial x^k}\left(\sum_a G_a(\psi_t|\Delta_a) - G_0\right), \tag{36}$$

where $n^*=n\sigma^D$. The generating functions are

$$G_a(\psi_t|\Delta_a) = -\frac{1}{2}\pi^{-1/2}S_D(1-z_1x)^{-(1/2)D}\left(\frac{1-z_1x}{2-x-z_2-z_1+xz_1z_2}\right)^{1/2}\int_0^\infty du K_a^*(\sqrt{u})\exp\left(\frac{(2-z_2-z_1)x}{2-x-z_2-z_1+xz_1z_2}\frac{1}{2}\Delta_a^*(\sqrt{u})\right)$$
$$\times \exp\left(-\frac{1-z_2x}{2-x-z_2-z_1+xz_1z_2}u\right)\exp\left(-\frac{1}{2}\frac{(z_2-z_1)x}{2-x-z_2-z_1+xz_1z_2}[u-\sqrt{u}\sqrt{u-2\Delta_a^*(\sqrt{u})}]\right) \tag{37}$$

and

$$G_0 = -\tfrac{1}{2}\pi^{-1/2}S_D(1-z_1x)^{-(D+1)/2}(2-x-z_2-z_1+xz_1z_2)^{1/2} \tag{38}$$

with S_D the area of the D-dimensional unit hard sphere,

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{39}$$

which is, e.g., 4π in three dimensions and 2π for $D=2$. The scaled probabilities and energy loss functions are

$$K^*(x) = K\left(x\sqrt{\frac{2}{m\beta}}\right),$$

$$\Delta^*(x) = \beta\Delta\left(x\sqrt{\frac{2}{m\beta}}\right). \tag{40}$$

In general, $K^*(x)$ and $\Delta^*(x)$ are functions of temperature and, hence, time but in order to keep the resulting expressions below from becoming too cumbersome, these arguments will be suppressed. The utility of this generating function, which is admittedly complex, is that the limits and derivatives needed to evaluate Eq. (36) are easily programmed using symbolic manipulation packages.

To complete the description of the uniform fluid, this paper gives the quantities necessary to calculate the lowest order correction. These are written conveniently as

$$I_{k,rs} = n^*\chi\frac{S_D}{2D(D+2)\sqrt{\pi}}\left(\frac{k_BT}{m\sigma^2}\right)^{1/2} I_{k,rs}^*,$$

$$I_{k,rs}^* = I_{k,rs}^{*E} + \sum_a \int_0^\infty K_a^*(\sqrt{u})I_{k,rs}^{*I}e^{-1/2u}du, \tag{41}$$

where $I_{02,2}^{*E}+I_{20,2}^{*E}=-8(D-1)$ and all other elastic contributions are zero, and the inelastic kernals are

$$I_{1,00}^{*I} = (D+2)\Delta^*(\sqrt{u}),$$

$$I_{2,00}^{*I} = [\Delta^*(\sqrt{u})+3-u]\Delta^*(\sqrt{u}),$$

$$I_{1,02}^{*I} + I_{1,20}^{*I} = \frac{D+2}{16}(u^2-6u+3)\Delta^*(\sqrt{u}),$$

$$I_{02,2}^{*I} + I_{20,2}^{*I} = \tfrac{1}{16}\{\Delta^*(\sqrt{u})[\Delta^*(\sqrt{u})(u^2-6u+3) - u^3 + 9u^2 - (8D+49)u + 8D + 37] + 16(D-1)(u-\sqrt{u}\sqrt{[u-2\Delta^*](\sqrt{u})})\}. \tag{42}$$

The pressure can similarly be expressed in the generating function formalism, but here we just give for later use the expression for the pressure (see Appendix B) including the lowest order corrections to the Gaussian distribution

$$\frac{p^{(0)}}{nk_BT} = 1 + n^*\chi\frac{S_D}{2D} + nk_BTn^*\chi\frac{S_D}{2D}\frac{1}{\sqrt{2\pi}} \times\sum_a\int_0^\infty vK_a^*(-v)g(v,\Delta_a^*(-v)) \times\left(1+\frac{D}{16}c_2(v^4-6v^2+3)\right)\exp\left(-\frac{1}{2}v^2\right)dv, \tag{43}$$

where the function $g(v,\Delta)$ is defined as

$$g(v,\Delta) = \mathrm{sgn}(v)\sqrt{v^2-2\Delta} - v. \tag{44}$$

C. The simple granular fluid in D dimensions

For a simple granular fluid with constant coefficient of restitution one has $\Delta^*(v)=\beta\Delta(v\sqrt{2k_BT/m})=(1-\alpha^2)\frac{1}{2}v^2$ and $K^*(x)=1$ giving

$$G(\psi_t|\Delta) \to -\pi^{-1/2}S_D(1-z_1x)^{-(1/2)D}(1-z_1x)^{1/2} \times(2-x-z_2-z_1+xz_1z_2)^{1/2} \times\left[-\tfrac{1}{2}(2-z_2-z_1)x(1-\alpha^2) + 2 - 2z_2x + (z_2-z_1)x(1-\alpha)\right]^{-1} \tag{45}$$

and

$$I_{1,00}^* = 2(D+2)(1-\alpha^2),$$

$$I_{2,00}^* = 2(1-2\alpha^2)(1-\alpha^2),$$

$$I_{1,02}^* + I_{1,20}^* = \frac{3(D+2)}{8}(1-\alpha^2),$$

$$I_{2,02}^{*}+I_{2,20}^{*}=-8(D-1)+\tfrac{1}{8}(\alpha-1)(30\alpha^{3}+30\alpha^{2}+24D\alpha+105\alpha+137-8D)$$
$$=\tfrac{1}{8}(\alpha+1)(30\alpha^{3}-30\alpha^{2}+24D\alpha+105\alpha-56D-73), \tag{46}$$

so that the lowest order correction to the Gaussian is

$$c_{2}=\frac{I_{2,00}}{-2I_{1,00}-(I_{2,20}+I_{2,02})}=\frac{16(1-2\alpha^{2})(1-\alpha)}{24D+9-(41-8D)\alpha+30\alpha^{2}(1-\alpha)}. \tag{47}$$

The cooling rate is

$$\xi^{(0)}=-(1-\alpha^{2})n^{*}\chi\frac{S_{D}}{2\sqrt{\pi}}\left(\frac{k_{B}T}{m\sigma^{2}}\right)^{1/2}nk_{B}T\left(1+\frac{3}{16}c_{2}\right) \tag{48}$$

and for a simple granular fluid, $g(v,\Delta^{*}(-v))=v(\alpha-1)$ gives

$$p^{(0)}=nk_{B}T\left(1+n^{*}\chi\frac{S_{D}}{4D}(1+\alpha)\right), \tag{49}$$

so it is seen that the second order terms do not contribute. Equations (47) and (48) agree with the results previously given for the simple granular gas by van Noije and Ernst [32].

This completes the discussion of the zeroth order Chapman-Enskog solution which is also the HCS. The simplest approximation is to take the distribution to be Maxwellian with the temperature obeying Eq. (22). The usual approximation is to keep c_2 while neglecting terms of order c_2^2 and c_k for $k>2$ resulting in Eq. (35) with coefficients that depend on the energy loss model.

IV. CHAPMAN-ENSKOG AT FIRST ORDER: THE NAVIER-STOKES EQUATIONS

The first order equation can be written as

$$\partial_{t}^{(0)}f_{1}-\mathcal{L}_{0}[f_{1}]=J_{1}[f_{0},f_{0}]-(\partial_{t}^{(1)}+\vec{v}\cdot\vec{\nabla})f_{0}, \tag{50}$$

where $\mathcal{L}_0[f_1]=(J_0[f_1,f_0]+J_0[f_0,f_1])$ is the linearized Boltzmann operator. The first order balance equations are used to eliminate the time derivative on the right-hand side. It is convenient to divide the first order heat source into two parts

$$\xi_{1}=\xi_{0}[f_{1}]+\xi_{1}[f_{0}], \tag{51}$$

where the first term on the right-hand side is, as indicated, a linear operator acting on the first order distribution and the second is of first order in the gradients and depends solely on the zeroth order distribution. Furthermore, since it is a scalar, $\xi_1[f_0]$ must be proportional to the only scalar gradient, namely $\vec{\nabla}\cdot\vec{u}$ so that we will write $\xi_1[f_0]=\xi_1^{\vec{\nabla}u}[f_0]\vec{\nabla}\cdot\vec{u}$ (see Appendix B for more details). Then, the first order equation becomes

$$\partial_{t}^{(0)}f_{1}+\frac{2}{Dnk_{B}T}\xi_{0}[f_{1}]\left(T\frac{\partial}{\partial T}f_{0}\right)-\mathcal{L}_{0}[f_{1}]=J_{1}[f_{0},f_{0}]-\frac{2}{Dnk_{B}T}\xi_{1}^{\vec{\nabla}u}[f_{0}](\vec{\nabla}\cdot\vec{u})\left(T\frac{\partial}{\partial T}f_{0}\right)-\sum_{\alpha}\sum_{i}B_{i}^{\alpha}(\vec{V};[f_{0}])\partial_{i}\psi_{t,\alpha} \tag{52}$$

with

$$B_{i}^{n}=\left(n^{-1}f_{0}+\frac{1}{nk_{B}T}\frac{\partial p^{(0)}}{\partial n}\left[\frac{\partial}{\partial z_{1}}f_{0}\right]_{T}\right)V_{1i},$$

$$B_{i}^{T}=\left(\frac{1}{nk_{B}T}\frac{\partial p^{(0)}}{\partial T}\left[\frac{\partial}{\partial z_{1}}f_{0}\right]_{T}+\frac{\partial}{\partial T}f_{0}\right)V_{1i},$$

$$B_{i}^{u_{j}}=\frac{2}{Dnk_{B}}\left(-p^{(0)}\frac{\partial}{\partial T}f_{0}-\frac{mnV_{1}^{2}}{2T}\left[\frac{\partial}{\partial z_{1}}f_{0}\right]_{T}-\frac{Dnk_{B}}{2}f_{0}\right)\delta_{ij}+\frac{m}{2k_{B}T}\left(V_{1i}V_{1j}-\frac{1}{D}V_{1}^{2}\delta_{ij}\right)\left[-\frac{\partial}{\partial z_{1}}f_{0}\right]_{T}, \tag{53}$$

where the variable $z=(m/2k_BT)V^2$.

It is shown in Appendix A that $J_1[f_0,f_0]$ can be written as

$$J_{1}[f_{0},f_{0}]=\sum_{\gamma,i}[\partial_{i}\psi_{t,\gamma}(\vec{r})]\left(J_{i}^{(0)}\left[f_{0},\frac{\partial}{\partial\psi_{\gamma}}f_{0}\right]+\frac{1}{2}\delta_{\gamma n}\frac{\partial\ln\chi}{\partial n}J_{i}^{(0)}[f_{0},f_{0}]\right), \tag{54}$$

where the detailed form of the operator $J_i^{(0)}$ is given in Appendix A. The right-hand side of Eq. (52) is therefore expressed entirely in terms of the gradients of the hydrodynamic fields so that the first order correction to the distribution must also be proportional to the gradients. Since the only vector available is $\vec{V}$ and the only tensors are the unit tensor and the symmetric traceless tensor $V_iV_j-(1/D)\delta_{ij}V^2$, the first order distribution must take the form

$$f_{1}=f_{0}(x_{1})\left[A^{(n)}(\vec{V}_{1})V_{1i}\partial_{i}n+A^{(T)}(\vec{V}_{1})V_{1i}\partial_{i}T+A^{(\vec{\nabla}u)}(\vec{V}_{1})\vec{\nabla}\cdot\vec{u}+\sqrt{\frac{D}{D-1}}A^{(\partial u)}(\vec{V}_{1})\left(V_{1i}V_{1j}-\frac{1}{D}\delta_{ij}V^{2}\right)\times\left(\partial_{i}u_{j}+\partial_{j}u_{i}-\frac{2}{D}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right)\right]. \tag{55}$$

Then, both sides of the kinetic equation are expressed in terms of the gradients of the hydrodynamic fields and since those gradients can vary independently, their coefficients must vanish individually giving

$$\phi_{I_{\alpha}}^{\alpha}(\vec{V}_{1})\left(\frac{2}{Dnk_{B}T}\xi_{0}[f_{0}]\frac{\partial}{\partial T}f_{0}A^{(\alpha)}+\sum_{\gamma}K_{\gamma}^{\alpha}[A^{(\gamma)}]\right)-\mathcal{L}_{0}[f_{0}A^{(\alpha)}\phi_{I_{\alpha}}^{\alpha}]=\Omega_{I_{\alpha}}^{\alpha}[f_{0},f_{0}]-C^{\alpha}(\vec{V}|f_{0})\phi_{I_{\alpha}}^{\alpha}(\vec{V}_{1}), \tag{56}$$

where greek indices range over the four values n, T, $\vec{\nabla}u$, and ∂u. In this equation, the capitalized index, I_α, is a superindex corresponding to a set of Cartesian indices as illustrated by the definition

$$\phi^{\alpha}_{I_\alpha}(\vec{V}) = \left(V_i, V_i, 1, \sqrt{\frac{D}{D-1}}\left(V_i V_j - \frac{1}{D} V^2 \delta_{ij} \right) \right). \quad (57)$$

The linear functional K^{α}_{γ} encapsulates contributions coming from the action of the functional derivative on the nonlocal term $\vec{\nabla}T$ as well as terms coming from $\xi^{(0)}[f_1]$ and is given by

$$K^{\alpha}_{\gamma}[A^{(\gamma)}] = A^{(T)} f_0 \left(\delta_{\alpha n} \frac{\partial}{\partial n} + \delta_{\alpha T} \frac{\partial}{\partial T} \right) \left(\frac{2}{D n k_B T} \xi_0[f_0] \right) - \delta_{\alpha \vec{\nabla} u} \frac{1}{D n k_B T} \xi_0[f_0](f_0 A^{(\vec{\nabla}u)}) \frac{\partial}{\partial T} f_0. \quad (58)$$

The source terms on the right-hand side are, after some manipulation, given by

$$C^{n}(\vec{V}|f_0) = n^{-1} f_0 + \frac{1}{n k_B T} \frac{\partial p^{(0)}}{\partial n} \left(\frac{\partial}{\partial z} f_0 \right)_T,$$

$$C^{T}(\vec{V}|f_0) = \frac{1}{n k_B T} \frac{\partial p^{(0)}}{\partial T} \left(\frac{\partial}{\partial z} f_0 \right)_T + \frac{\partial}{\partial T} f_0,$$

$$C^{\vec{\nabla}u}(\vec{V}|f_0) = \frac{2}{D n k_B} (\xi_1^{\vec{\nabla}u} - p^{(c)}) \frac{\partial f_0}{\partial T} - \frac{2}{D} T \frac{\partial c_j}{\partial T} \frac{\partial f_0}{\partial c_j},$$

$$C^{\partial u}(\vec{V}|f_0) = -\frac{m}{2 k_B T} \left(\frac{\partial}{\partial z} f_0 \right)_T \quad (59)$$

and

$$\Omega^{n}_{i}[f_0, f_0] = \frac{1}{2} \frac{\partial \ln n^2 \chi}{\partial n} J^{(0)}_i [f_0, f_0],$$

$$\Omega^{T}_{i}[f_0, f_0] = J^{(0)}_i \left[f_0, \frac{\partial}{\partial T} f_0 \right],$$

$$\Omega^{\vec{\nabla}u}[f_0, f_0] = \frac{1}{D} \sum_i J^{(0)}_i \left[f_0, \frac{\partial}{\partial u_i} f_0 \right],$$

$$\Omega^{\partial u}_{ij}[f_0, f_0] = \frac{1}{4} \left(J^{(0)}_j \left[f_0, \frac{\partial}{\partial u_i} f_0 \right] + J^{(0)}_i \left[f_0, \frac{\partial}{\partial u_j} f_0 \right] - \frac{2}{D} \delta_{ij} \sum_l J^{(0)}_l \left[f_0, \frac{\partial}{\partial u_l} f_0 \right] \right). \quad (60)$$

We conclude the discussion of the first order approximation with some general remarks concerning the solution of Eqs. (56)–(60). First, the hydrodynamic fields are defined by Eq. (11) which can be written as

$$\psi_{t,i}(\vec{r}) = \int \hat{\psi}_i(\vec{V}) f(\vec{r}, \vec{V}; t) d\vec{V} \quad (61)$$

with the array of velocity moments $\hat{\psi}(\vec{V}) = (1, (m/2)V^2, \vec{V})$. However, from the definition Eq. (26), it is clear that the zeroth order distribution satisfies

$$\psi_{t,i}(\vec{r}) = \int \hat{\psi}_i(\vec{V}) f_0(\vec{r}, \vec{V}; t) d\vec{V} \quad (62)$$

so that it must be the case that all higher order contributions to the distribution give

$$0 = \int \hat{\psi}_i(\vec{V}) f_j(\vec{r}, \vec{V}; t) d\vec{V} \quad (63)$$

for all i and j. Since the gradients of the hydrodynamic fields are arbitrary, this means that in the case of the first order distribution, the coefficients of the gradients must be orthogonal to the first three velocity moments under the measure $f_0(\vec{V})$ or

$$0 = \int \hat{\psi}_i(\vec{V}) A^{(\alpha)}(\vec{V}) f_0(\vec{r}, \vec{V}; t) d\vec{V}. \quad (64)$$

Second, it is clear that Eq. (56) is a linear equation in the coefficients $A^{(\alpha)}(\vec{V})$ so that the conditions for the existence of a solution follows the usual theory of linear operators. In particular, defining a Hilbert space with measure $f_0(\vec{V})$ the Fredholm alternative, which states that for linear operator L and source term B, the equation $LV=B$ has a solution if and only if B is orthogonal to the null space of L. We expect that $\hat{\psi}(\vec{V})$ is in the null space of the operator defined by the left-hand side of Eq. (56). In fact, it is clear that multiplying by $\hat{\psi}_i(\vec{V}_1)$ and integrating over velocities gives, on the left-hand side,

$$-\delta_{\alpha \vec{\nabla} u} \frac{1}{D n k_B T} \xi_0(f_0 A^{(\vec{\nabla}u)}) \frac{\partial}{\partial T} \int d\vec{V}_1 \hat{\psi}_i(\vec{V}) \phi^{\alpha}_{I_\alpha}(\vec{V}_1) f_0 - \int d\vec{V}_1 \hat{\psi}_i(\vec{V}_1) \mathcal{L}_0(f_0 A^{(\alpha)} \phi^{\alpha}_{I_\alpha}). \quad (65)$$

Now, the $\mathcal{L}_0$ term vanishes for $\hat{\psi}_i = 1$ and $\vec{V}$ due to the conservation of particle number and total momentum, respectively. For the last choice, $\hat{\psi}_i = (m/2)V^2$, it is only nonzero for $\alpha = \vec{\nabla}u$ due to rotational symmetry (for other choices of α, $\phi^{\alpha}_{I_\alpha}$ is a vector or traceless tensor). Thus the only nonvanishing element of this system of equations is that for $\alpha = \vec{\nabla}u$ and $\hat{\psi}_i = (m/2)V^2$ which becomes

$$-\frac{1}{2T} \xi_0(f_0 A^{(\vec{\nabla}u)}) - \int d\vec{V}_1 \frac{m}{2} V^2 \mathcal{L}_0(f_0 A^{(\vec{\nabla}u)}) \quad (66)$$

and which is seen to vanish from the definition of $\xi_0[g]$, Eq. (B23) and $\mathcal{L}_0[g]$. Thus, $\hat{\psi}$ is indeed in the null space of the linear operator and a necessary condition for the existence of a solution is that the right-hand side is orthogonal to $\hat{\psi}$ as well. That this is indeed the case is easily verified.

A. Approximate solution of the integral equations

The integral equations summarized by Eqs. (56)–(60) will be solved by expanding the unknown functions $A^{(\gamma)}$ in associated Laguerre polynomials as

$$A^{(\alpha)}(\vec{V}) = \sum_s a_s^{(\alpha)} L_s^{\{[(D-2/2)]+\lambda_\alpha\}}\left(\frac{m}{2k_BT}V^2\right), \tag{67}$$

where the coefficients are in general functions of the hydrodynamic fields, $a_s^{(\alpha)}=a_s^{(\alpha)}(\psi_t)$ although, for clarity, this dependence will be suppressed below. It is interesting to note that for a simple granular gas, Garzo and Dufty [6] wrote the first order distribution as in Eq. (55) but with f_0 replaced by the Maxwellian $f_M(v_1;\psi_t)$. The use of f_0 here is motivated by the fact that the source term in the first order equations, Eq. (56), is proportional to f_0 so that it seems appropriate to use this in the definition of the first order correction but this is not necessary. Of course, if the various expansions are convergent, then they must be equivalent and one clearly has that

$$\begin{aligned} f_0(\vec{V})A^{(\alpha)}(\vec{V}) &= f_0(\vec{V})\sum_s a_s^{(\alpha)} L_s^{[(D-2)/2+\lambda_\alpha]}\left(\frac{m}{2k_BT}V^2\right) \\ &= f_M(V;\psi_t)\sum_s \bar{a}_s^{(\alpha)} L_s^{[(D-2)/2+\lambda_\alpha]}\left(\frac{m}{2k_BT}V^2\right) \end{aligned} \tag{68}$$

for coefficients $\bar{a}_s^{(\alpha)}$ which are linear combinations of the $a_s^{(\alpha)}$ and in particular, if $a_{s_0}^{(\alpha)}$ is the first nonvanishing coefficient in the sum, then $a_{s_0}^{(\alpha)}=\bar{a}_{s_0}^{(\alpha)}$. This seems to suggest that the ordering of the polynomial expansion of f_0, given in Eq. (26) and that of the functions $A^{(\alpha)}$ introduced here are not independent since terms of order c_2 in f_0 are compensated for by terms of order $a_{s_{0+2}}^{(\alpha)}$ in Eq. (68). If we think of these expansions of $f_0(\vec{V})$ in Eq. (26) as being perturbative expansions in a fictitous parameter ϵ so that $c_n\sim\epsilon^n$ then this seems to imply that we should consider the terms in Eq. (67) above to be ordered by the same parameter so that $a_s^{(\alpha)}\sim\epsilon^{s+s_\alpha}$ for some constant s_α. Regardless of the actual value of s_α, this implies that terms of order $a_s^{(\alpha)}c_2$ are of higher order in ϵ than are terms of order $a_s^{(\alpha)}$ and should be treated accordingly in the perturbative expansion. In particular, at lowest order in ϵ, terms of order $a_{s_0}^{(\alpha)}c_2$ should therefore be neglected. One consequence of this is that the exact property alluded to above, namely that $a_{s_0}^{(\alpha)}=\bar{a}_{s_0}^{(\alpha)}$, is automatically preserved by the perturbative expansions. In the development to follow, the usual approximation will be made wherein we work to lowest order in perturbation theory so that only the lowest nonvanishing coefficient is retained (the "lowest Sonine approximation"). Based on the present discussion, all terms of order $a_{s_0}^{(\alpha)}c_2$ will be neglected (as well as terms of order $c_2^2, a_{s_0}^{(\alpha)}a_{s_0}^{(\beta)}$, etc.). In the Conclusions, the present approximation is evaluated for the special case of a simple granular fluid in three dimensions by comparison to expressions given in Ref. [6] where all dependence on c_2 is retained.

In order to solve the integral equations, the expansions Eq. (67) are substituted into Eq. (56) and the αth equation is multiplied by $\phi_{I_\alpha}^{\alpha}(\vec{V}_1)L_k^{[(D-2)/2+\lambda_\alpha]}[(m/2k_BT)V_1^2]$ and tensorial indices contracted and $\vec{V}_1$ integrated. The left-hand side of these equations is found to be simplified by the choices $\lambda_n=\lambda_T=1$, $\lambda_{\nabla u}=0$, and $\lambda_{\partial u}=2$ which are made henceforth. The result after some simplification can be written as

$$\begin{aligned} &\frac{2}{Dk_B}\xi_0\frac{\Gamma\left(\frac{D}{2}+\lambda_\alpha+k\right)}{\Gamma(D/2)\Gamma(k+1)}\left(\frac{2k_BT}{m}\right)^{\lambda_\alpha}\left(\frac{\partial}{\partial T}a_k^{(\alpha)}+\frac{1}{T}(\lambda_\alpha+k)a_k^{(\alpha)}\right. \\ &\quad\left.-\frac{1}{T}ka_{k-1}^{(\alpha)}\right)+\sum_l I_{kl}^{\alpha}a_l^{(\alpha)}-\sum_l K_{kl}^{\alpha\alpha'}a_l^{(\alpha')}=\Lambda_k^{\alpha}+\Omega_k^{\alpha}, \end{aligned} \tag{69}$$

where the contributions from the Boltzmann operator are

$$\begin{aligned} I_{kl}^{\alpha} = -\sum_{I_\alpha}\int d\vec{V}_1\phi_{I_\alpha}^{\alpha}(\vec{V}_1)L_k^{[(D-2)/2+\lambda_\alpha]}\left(\frac{m}{2k_BT}V_1^2\right) \\ \times\mathcal{L}_0\left[L_l^{[(D-2)/2+\lambda_\alpha]}\left(\frac{m}{2k_BT}V^2\right)\phi_{I_\alpha}^{\alpha}(\vec{V})\right], \end{aligned} \tag{70}$$

and the last coefficient on the left-hand side is

$$\begin{aligned} K_{kl}^{\alpha\alpha'} = \int d\vec{V}_1\phi_{I_\alpha}(\vec{V}_1)L_k^{[(D-2)/2+\lambda_\alpha]}\left(\frac{m}{2k_BT}V_1^2\right) \\ \times K_{\alpha'}^{\alpha}\left[\phi_{I_{\alpha'}}(\vec{V}_1)L_l^{[(D-2)/2+\lambda_{\alpha'}]}\left(\frac{m}{2k_BT}V^2\right)\right]. \end{aligned} \tag{71}$$

The source terms on the right-hand side are

$$\Lambda_k^{\alpha} = -\int d\vec{V}_1\phi_{I_\alpha}(\vec{V}_1)L_k^{[(D-2)/2+\lambda_\alpha]}\left(\frac{m}{2k_BT}V_1^2\right)C^{\alpha}(\vec{V}|f_0)\phi_{I_\alpha}(\vec{V}_1) \tag{72}$$

and

$$\Omega_k^{\alpha} = \int d\vec{V}_1\phi_{I_\alpha}(\vec{V}_1)L_k^{[(D-2)/2+\lambda_\alpha]}\left(\frac{m}{2k_BT}V_1^2\right)\Omega_i^{\alpha}[f_0,f_0]. \tag{73}$$

A straightforward evaluation using the orthogonality and standard recursion relations of the associated Laguerre polynomials [33] gives

$$\begin{aligned} K_{kl}^{\alpha\alpha'} &= \frac{\Gamma\left(\frac{1}{2}D+k+1\right)}{\Gamma\left(\frac{1}{2}D\right)\Gamma(k+1)}n\left(\frac{2k_BT}{m}\right)\delta_{\alpha' T}\delta_{kl} \\ &\quad\times\left(\delta_{\alpha n}\frac{\partial}{\partial n}+\delta_{\alpha T}\frac{\partial}{\partial T}\right)\frac{2}{Dnk_B}\xi_0+\delta_{k1}\delta_{\alpha\vec{\nabla}u}\delta_{\alpha'\vec{\nabla}u} \\ &\quad\times\frac{1}{2nk_BT}\xi^{(0)}\left[f_0L_l^{[(D-2)/2]}\left(\frac{m}{2k_BT}V^2\right)\right]\left(\frac{1}{T}\right)n \end{aligned} \tag{74}$$

and

$$\Lambda_k^{\alpha} = \frac{\Gamma\left(\frac{D-2}{2}+k+1\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma(k+1)} \left\{ \begin{array}{c} \frac{2k_BT}{m}\left(\frac{1}{2}D+k\right)\left(\frac{1}{k_BT}\frac{\partial p^{(0)C}}{\partial n}c_k + c_{k+1}\right) \\ -\frac{n}{T}\frac{2k_BT}{m}\left(\frac{1}{2}D+k\right)\left[\left(-\frac{1}{nk_BT}T\frac{\partial p^{(0)}}{\partial T}+2k+1\right)c_k-(k+1)c_{k+1}-kc_{k-1}+\frac{\partial c_k}{\partial T}-\frac{\partial c_{k+1}}{\partial T}\right] \\ \frac{2}{Dnk_BT}(\xi_1^{\nabla u}[f_0]-p^{(0)c})k(c_{k-1}-c_k)-\frac{2}{Dnk_BT}(\xi_1^{\vec{\nabla}u}[f_0]-p^{(0)})T\frac{\partial c_k}{\partial T} \\ \frac{2nk_BT}{m}\sqrt{\frac{D-1}{D}}\frac{(D+k)(D+k+2)}{4}(c_{k+1}-c_k) \end{array} \right\}. \quad (75)$$

Finally, it is useful to note that the lowest order coefficients are related to the kinetic parts of the transport coefficients (see Appendix B). Specifically, the first order contribution to the pressure tensor takes the usual form

$$P_{ij}^{(1)} = -\eta\left(\partial_i u_j + \partial_j u_i - \frac{2}{D}\delta_{ij}(\vec{\nabla}\cdot\vec{u})\right) - \gamma\delta_{ij}(\vec{\nabla}\cdot\vec{u}), \quad (76)$$

where the shear viscosity, η, has a kinetic contribution given by

$$\eta^K = -2nk_BT\left(\frac{k_BT}{m}\right)\sqrt{\frac{D}{D-1}}a_0^{\partial u}, \quad (77)$$

and the bulk viscosity γ has no kinetic contribution. The first order contribution to heat flux vector is

$$\vec{q}^{(1)}(\vec{r},t) = -\mu\vec{\nabla}\rho - \kappa\vec{\nabla}T, \quad (78)$$

where κ is the coefficient of thermal conductivity and μ is a transport coefficient characterizing the way in which density gradients can cause heat flow due to differential cooling rates. It vanishes in the elastic limit. The kinetic parts of these transport coefficients are given by

$$\mu^K = \left[nk_BT\left(\frac{k_BT}{m}\right)\frac{D+2}{2}\right]a_1^{\rho},$$

$$\kappa^K = \left[nk_BT\left(\frac{k_BT}{m}\right)\frac{D+2}{2}\right]a_1^{T}. \quad (79)$$

These expressions are exact if f_0 is replaced by a Gaussian in Eq. (55) but if the first order correction is written in terms of f_0 then there are terms in c_2 which would contribute (as discussed in Appendix B) in principle but which would in any case be dropped here since they are of order $c_2 a_{s_0}^{(\gamma)}$. The collisional contributions will be discussed below.

B. Lowest order approximations

The simplest nontrivial approximation is to keep only the lowest order nonzero coefficient in each expansion in Eq. (67). This means $a_1^{(n)}$, $a_1^{(T)}$, $a_2^{(\vec{\nabla}u)}$, and $a_0^{(\partial u)}$. Since the transport coefficients are more interesting than the distribution itself, we write these equations in terms of the kinetic parts of the transport coefficients giving

$$\xi_0(D+2)\frac{1}{m}T\frac{\partial\mu^K}{\partial T} + I_{11}^n\mu^K - \xi_0(D+2)\frac{1}{m}T\left(\frac{\partial \ln n\chi_0}{\partial n}\right)\kappa^K = \left[nk_BT\left(\frac{k_BT}{m}\right)\frac{D+2}{2}\right]\left(\Omega_1^n + \frac{k_BT}{m}\frac{D(D+2)}{2}c_2\right),$$

$$\xi_0(D+2)\frac{1}{m}\left[T\frac{\partial\kappa^K}{\partial T} + \left(T\frac{\partial}{\partial T}\ln\xi_0\right)\kappa^K\right] + I_{11}^T\kappa^K = mn\left(\frac{k_BT}{m}\right)^2\frac{D+2}{2}\left[\Omega_1^T + \frac{n}{T}\frac{k_BT}{m}\frac{D(D+2)}{2}\times\left(1+2c_2+\frac{\partial c_2}{\partial T}\right)\right],$$

$$\frac{1}{4}\xi_0\frac{D+2}{k_BT}\left(T\frac{\partial}{\partial T}a_2^{(\vec{\nabla}u)} + 2a_2^{(\vec{\nabla}u)}\right) + I_{22}^{\vec{\nabla}u}a_2^{(\vec{\nabla}u)} = \Omega_2^{\vec{\nabla}u} - \frac{D+2}{2k_BT}(\xi_1^{\vec{\nabla}u}[f_0]-p^{(0)c})c_2 - \frac{D+2}{4k_BT}(\xi_1^{\vec{\nabla}u}[f_0]-p^{(0)})T\frac{\partial c_2}{\partial T},$$

$$\xi_0\frac{D+2}{m}\left(\frac{2k_BT}{m}\right)T\frac{\partial\eta^K}{\partial T} + I_{00}^{\partial u}\eta^K = mn\left(\frac{k_BT}{m}\right)^2\left[n\left(\frac{k_BT}{m}\right)D(D+2) - 2\sqrt{\frac{D}{D-1}}\Omega_0^{\partial u}\right]. \quad (80)$$

The Boltzmann integrals and the source terms are evaluated in a straightforward manner and the present evaluations were performed as described in Appendix D, making frequent use of symbolic manipulation. Using the definitions of the functions $g(v)$ [Eq. (44)] and $K_a^*(v)$ and $\Delta_a^*(v)$ given in Eq. (40), the results can be written as

$$I_{rs}^{\gamma} = I_{rs}^{\gamma E}\left[1 + \sum_a \int_0^{\infty} K_a^*(-v)e^{-1/2v^2}v\left(\Delta_a^*(-v)S_{rs}^{\gamma}(v) + \frac{1}{4}vg(v,\Delta_a^*(-v))\right)dv\right],$$

$$\Omega_{rs}^{\gamma} = \Omega_{rs}^{\gamma E} + \chi n^2 \sigma^D S_D \frac{1}{\sqrt{2\pi}} \sum_a \int_0^\infty K_a^*(-v) \times e^{-1/2v^2} v \left(T_{rs}^{\gamma}(v) + U_{rs}^{\gamma}(v) c_2 + V_{rs}^{\gamma}(v) \frac{dc_2}{dT} \right) dv, \tag{81}$$

with the elastic contributions

$$I_{11}^{nE} = I_{11}^{TE} = n^2 \sigma^{D-1} S_D \chi \left(\frac{k_B T}{m} \right)^{3/2} \frac{2(D-1)}{\sqrt{\pi}},$$

$$I_{22}^{\vec{\nabla} u E} = n^2 \sigma^{D-1} S_D \chi \left(\frac{k_B T}{m} \right)^{1/2} \frac{(D-1)}{2\sqrt{\pi}},$$

$$I_{00}^{\partial u E} = \chi n^2 \sigma^{D-1} S_D \left(\frac{k_B T}{m} \right)^{5/2} \frac{4D}{\sqrt{\pi}} \tag{82}$$

and the inelastic kernals

$$S_{11}^{n}(v) = S_{11}^{T}(v) = \frac{D+8}{16(D-1)} (v^2 - 1),$$

$$S_{22}^{\vec{\nabla} u}(v) = \frac{1}{64(D-1)} [v^6 - 9v^4 + (8D+49)v^2 - 37 - 8D] - \frac{1}{64(D-1)} (v^4 - 6v^2 + 3) \Delta^*(v),$$

$$S_{00}^{\partial u}(v) = \frac{1}{4D} (v^2 - 1). \tag{83}$$

The elastic contributions to the sources are

$$\Omega_1^{nE} = \frac{1}{2} \frac{\partial n^2 \chi}{\partial n} \sigma^D S_D \left(\frac{k_B T}{m} \right) \frac{D+5}{4} c_2,$$

$$\Omega_1^{TE} = n^2 \sigma^D \chi S_D \frac{k_B T}{m} \frac{1}{T} \frac{3}{4} \left(1 + 2c_2 + \frac{dc_2}{dT} \right),$$

$$\Omega_1^{\vec{\nabla} u E} = n^2 \sigma^D \chi S_D \frac{D-7}{8D} c_2,$$

$$\Omega_1^{\partial u E} = - n^2 \sigma^D \chi S_D \frac{k_B T}{m} \frac{1}{2} \sqrt{\frac{D-1}{D}}, \tag{84}$$

and the inelastic kernals are

$$T_{11}^{n}(v) = \frac{1}{2} \frac{\partial \ln n^2 \chi}{\partial n} \left(\frac{k_B T}{m} \right) \frac{1}{4} [(v^2 - 3) g(v, \Delta_a^*(-v)) - 2\Delta_a^*(v) [g(v, \Delta_a^*(-v)) + v]],$$

$$T_{11}^{T}(v) = \frac{k_B}{16m} \{ (v^4 - 4v^2 + 9) g(v, \Delta_a^*(-v)) - 2\Delta_a^*(-v) [(v^2 - 1) g(v, \Delta_a^*(-v)) + v^3 + 5v] \},$$

$$T_{22}^{\vec{\nabla} u}(v) = \frac{1}{8D} \{ [2\Delta_a^*(v) + 3 - v^2] g(v, \Delta_a^*(-v)) + v \Delta_a^*(-v) [v^2 - 1 - \Delta_a^*(-v)] \},$$

$$T_{00}^{\partial u}(v) = \frac{1}{2} \sqrt{\frac{D-1}{D}} \left(\frac{k_B T}{m} \right) [v \Delta_a^*(-v) - g(v, \Delta_a^*(-v))], \tag{85}$$

and

$$U_{11}^{n}(v) = \frac{1}{2} \frac{\partial \ln n^2 \chi}{\partial n} \left(\frac{k_B T}{m} \right) \frac{1}{64} [v^6 - 9v^4 + (49 + 8D) v^2 - 37 - 8D] g(v, \Delta_a^*(-v)) + \frac{1}{2} \frac{\partial \ln n^2 \chi}{\partial n} \left(\frac{k_B T}{m} \right) \frac{1}{32} \Delta_a^*(-v) \times (-v^4 + 6v^2 - 3) [g(v, \Delta_a^*(-v)) + v],$$

$$U_{11}^{T}(v) = - \frac{k_B}{m} \frac{1}{256} [-v^8 + 14v^6 + (-8D - 88) v^4 + (126 + 48D) v^2 - 24D + 33] g(v, \Delta_a^*(-v)) - \frac{k_B}{m} \frac{1}{128} \Delta_a^*(-v) [(v^6 - 11v^4 + 21v^2 - 3) \times g(v, \Delta_a^*(-v)) + v(v^6 - 5v^4 + 9v^2 - 57)],$$

$$U_{22}^{\vec{\nabla} u}(v) = \frac{1}{128D} v \Delta_a^*(-v) [\Delta_a^*(-v)(-v^4 + 10v^2 - 15) + v^6 - 11v^4 + v^2(61 + 8D) - 123 - 24D] + \frac{1}{128D} g(v, \Delta_a^*(-v)) [2\Delta_a^*(-v)(v^4 - 6v^2 + 3) - v^6 + 9v^4 + (8D - 65) v^2 - 8D + 53],$$

$$U_{00}^{\partial u}(v) = \sqrt{\frac{D-1}{D}} \left(\frac{k_B T}{m} \right) \frac{1}{32} [v \Delta_a^*(v^4 - 10v^2 + 15) - g(v, \Delta_a^*(-v))(v^4 - 6v^2 + 3)]. \tag{86}$$

The only nonvanishing coefficient of the temperature derivative is

$$V_{11}^{T}(v) = - \tfrac{1}{128} g(v, \Delta_a^*(-v))(-v^6 + 9v^4 - 57v^2 + 45) - \tfrac{1}{64} \Delta_a^*(-v) [(v^4 - 6v^2 + 3) g(v, \Delta_a^*(-v)) + (v^4 + 6v^2 - 33) v]. \tag{87}$$

The collisional contributions to the shear and bulk viscosity are, in this approximation,

$$\eta^C = \frac{2}{3} \theta \eta^K + \frac{D}{D+2} \gamma_1,$$

$$\gamma = \gamma_1 - (n k_B T) a_2^{\vec{\nabla} u} \frac{S_D n^* \chi}{32 D \sqrt{2\pi}} \sum_a \int_0^\infty K_a^*(-v) \times e^{-1/2v^2} v (3 - 6v^2 + v^4) g(v, \Delta_a^*(-v)) dv,$$

$$\mu^C = \theta\mu^K,$$

$$\kappa^C = \theta\kappa^K + \frac{D}{2}\frac{k_B}{m}\gamma_1 - m\chi n^2\sigma^{D+1}\left(\frac{k_BT}{m}\right)^{3/2} \times \frac{1}{4T\sqrt{\pi}}\frac{S_D}{D}c_2\left(1 + \frac{1}{4}\sum_a \int_0^\infty K_a^*(-v)v^2 \times e^{-(1/2)v^2}(v^2-3)g(v,\Delta_a^*(-v))dv\right) \tag{88}$$

with

$$\gamma_1 = m\sigma n\left(\frac{k_BT}{m}\right)^{1/2}\frac{S_Dn^*\chi}{D^2\sqrt{\pi}} \times \left[1 - \frac{1}{16}c_2 + \frac{1}{4}\sum_a \int_0^\infty K_a^*(-v)e^{-(1/2)v^2}v^2g(v,\Delta_a^*(-v)) \times \left(1 + \frac{1}{16}c_2(v^4 - 10v^2 + 15)\right)dv\right],$$

$$\theta = \frac{3S_D}{2D(D+2)}n^*\chi\left[1 + \frac{1}{2\sqrt{2\pi}}\sum_a \int_0^\infty K_a^*(-v)v \times e^{-(1/2)v^2}(v^2-1)g(v,\Delta_a^*(-v))dv\right]. \tag{89}$$

Finally, the first order corrections to the heat source are

$$\xi_0[f_1] = -(\vec{\nabla}\cdot\vec{u})a_2^{\vec{\nabla}u}n^2\sigma^D\chi S_D\left(\frac{k_BT}{m\sigma^2}\right)^{1/2}\frac{k_BT}{32\sqrt{\pi}}\sum_a \int_0^\infty K_a^*(-v)ve^{-(1/2)v^2}\Delta_a^*(-v)(v^4 - 6v^2 + 3)dv,$$

$$\xi_1[f_0] = (\vec{\nabla}\cdot\vec{u})n^2\sigma^D\chi S_D\frac{k_BT}{2\sqrt{2\pi}D}\sum_a \int_0^\infty K_a^*(-v)\Delta_a^*(-v)v^2e^{-(1/2)v^2}dv + c_2(\vec{\nabla}\cdot\vec{u})n^2\sigma^D\chi S_D\frac{k_BT}{32D\sqrt{2\pi}} \times \sum_a \int_0^\infty K_a^*(-v)\Delta_a^*(-v)e^{-(1/2)v^2}(15 - 10v^2 + v^4)v^2dv. \tag{90}$$

Equations (80)–(90) are the primary results of this paper. They give a prescription for the evaluation of the transport properties for an arbitrary model of energy dissipation at the Navier-Stokes level and in the usual, lowest Sonine approximation. In the next section, these are illustrated by using them to give the transport properties of a simple granular fluid.

C. Application: Transport in simple granular fluids

For the simple granular fluid, recall that $\Delta^*(v) = (1-\alpha^2)\frac{1}{2}v^2$ and $g(v,\Delta) = v(\alpha-1)$. Since there is no other energy scale, the coefficients of the first order solution must scale with temperature as

$$a_1^{(n)} \sim T^{-1/2},$$

$$a_1^{(T)} \sim T^{-3/2},$$

$$a_2^{(\nabla u)} \sim T^{-1/2},$$

$$a_0^{(\partial u)} \sim T^{-3/2},$$

$$\xi_0 \sim T^{3/2}, \tag{91}$$

giving

$$\left(\xi_0(D+2)\frac{3}{2m} + I_{11}^n\right)\mu^K + \xi_0(D+2)\frac{1}{m}T\left(\frac{\partial \ln n\chi_0}{\partial n}\right)\kappa^K = \left[nk_BT\left(\frac{k_BT}{m}\right)\frac{D+2}{2}\right]\left(\Omega_1^n + \frac{D(D+2)}{2}\frac{k_BT}{m}c_2\right),$$

$$\left(\xi_0(D+2)\frac{2}{m} + I_{11}^T\right)\kappa^K = mn\left(\frac{k_BT}{m}\right)^2\frac{D+2}{2}\Omega_1^T + \frac{1}{T}mn^2\left(\frac{k_BT}{m}\right)^3 \times \frac{D(D+2)^2}{4}(1+2c_2),$$

$$\left(\frac{3}{8}\xi_0\frac{D+2}{k_BT} + I_{22}^{\vec{\nabla}u}\right)a_2^{(\vec{\nabla}u)} = \Omega_2^{\vec{\nabla}u} - \frac{D+2}{2k_BT}(\xi_1^{\vec{\nabla}u}[f_0] - p^{(0)c})c_2,$$

$$\left[\xi_0\frac{D+2}{m}\left(\frac{k_BT}{m}\right) + I_{00}^{\partial u}\right]\eta^K = mn\left(\frac{k_BT}{m}\right)^2\left[n\left(\frac{k_BT}{m}\right)D(D+2) - 2\sqrt{\frac{D}{D-1}}\Omega_0^{\partial u}\right]. \tag{92}$$

From the zeroth order solution, one has

$$\xi_0[f_0] = -(1-\alpha^2)n^*\chi\frac{S_D}{2\sqrt{\pi}}\left(\frac{k_BT}{m\sigma^2}\right)^{1/2}nk_BT. \tag{93}$$

Equations (81) are easily evaluated giving the Boltzmann integrals

$$I_{11}^n = I_{11}^T = -n^2\sigma^{D-1}S_D\chi\left(\frac{k_BT}{m}\right)^{3/2}\frac{1}{8\sqrt{\pi}}(\alpha+1)\times[3\alpha(D+8)-11D-16],$$

$$I_{22}^{\vec{\nabla}u} = -n^2\sigma^{D-1}S_D\chi\left(\frac{k_BT}{m}\right)^{1/2}\frac{1}{128\sqrt{\pi}}(\alpha+1)(30\alpha^3-30\alpha^2+105\alpha+24\alpha D-56D-73),$$

$$I_{00}^{\partial u} = n^2\sigma^{D-1}S_D\chi\left(\frac{k_BT}{m}\right)^{5/2}\frac{1}{\sqrt{\pi}}(1+\alpha)(3-3\alpha+2D), \tag{94}$$

and sources

$$\Omega_1^n = \frac{1}{2}\frac{\partial n^2\chi}{\partial n}\sigma^D S_D\left(\frac{k_BT}{m}\right)\left(\frac{3}{8}(-1+\alpha^2)\alpha + \frac{1}{16}c_2(1+\alpha)(3\alpha^2-3\alpha+2D+10)\right),$$

$$\Omega_1^T = \frac{3}{16}n^2\sigma^D\chi S_D\frac{k_B}{m}(\alpha+1)^2[2\alpha-1+(\alpha+1)c_2],$$

$$\Omega_2^{\vec{\nabla}u} = \frac{3}{64D}n^2\sigma^D\chi S_D(1+\alpha)\left((5\alpha-1)(1-\alpha)(1+\alpha) - \frac{1}{6}c_2[15\alpha^3-3\alpha^2+3(4D+15)\alpha-(20D+1)]\right),$$

$$\Omega_0^{\partial u} = -\frac{1}{8}n^2\sigma^D\chi S_D k_BT\sqrt{\left(\frac{D-1}{D}\right)}(\alpha+1)\frac{3\alpha-1}{m}. \tag{95}$$

The low density (Boltzmann) transport coefficients in the elastic ($\alpha=1$) limit can be read off and are

$$\kappa_0 = k_B\left(\frac{k_BT}{m}\right)^{1/2}\frac{\sqrt{\pi}}{8\sigma^{D-1}S_D}\frac{D(D+2)^2}{D-1},$$

$$\eta_0 = \sqrt{\frac{mk_BT}{\pi}}\frac{(D+2)\pi}{4S_D}\sigma^{1-D},$$

$$\mu_0 = \gamma_0 = 0. \tag{96}$$

To facilitate comparison of the finite density transport coefficients to those of Ref. [6], it is useful to introduce dimensionless quantities

$$\nu_T^* = \chi\frac{D-1}{2D}(\alpha+1)\left(1+\frac{3(D+8)(1-\alpha)}{8(D-1)}\right),$$

$$\nu_\eta^* = \chi\left(1-\frac{1}{4D}(1-\alpha)(2D-3\alpha-3)\right),$$

$$\nu_\gamma^* = -\frac{1}{48}\chi\left(\frac{\alpha+1}{2}\right)(30\alpha^3-30\alpha^2+105\alpha+24\alpha D-56D-73),$$

$$\zeta^* = \frac{D+2}{4D}\chi(1-\alpha^2), \tag{97}$$

so that

$$I_{11}^n = I_{11}^T = \frac{2D}{\sqrt{\pi}}S_Dnn^*\left(\frac{k_BT}{m}\right)\left(\frac{k_BT}{m\sigma^2}\right)^{1/2}\nu_T^*,$$

$$I_{00}^{\partial u} = \frac{4D}{\sqrt{\pi}}S_Dnn^*\left(\frac{k_BT}{m}\right)^2\left(\frac{k_BT}{m\sigma^2}\right)^{1/2}\nu_\eta^*,$$

$$I_{22}^{\vec{\nabla}u} = \frac{3}{4\sqrt{\pi}}S_Dnn^*\left(\frac{k_BT}{m\sigma^2}\right)^{1/2}\nu_\gamma^*,$$

$$\frac{1}{m}\xi^{(0)} = -\frac{2D}{\sqrt{\pi}(D+2)}S_Dnn^*\frac{k_BT}{m}\left(\frac{k_BT}{m\sigma^2}\right)^{1/2}\zeta^*\left(1+\frac{3}{16}c_2\right), \tag{98}$$

and then

$$\kappa^K = \kappa_0\frac{D-1}{D}[\nu_T^*-2\zeta^*]^{-1}\left(1+2c_2+\frac{3}{8D(D+2)}n^*\chi S_D(\alpha+1)^2[2\alpha-1+(\alpha+1)c_2]\right),$$

$$\mu^K = 2\kappa_0\frac{T}{n}[2\nu_T^*-3\zeta^*]^{-1}\left[n\left(\frac{\partial \ln n\chi_0}{\partial n}\right)\kappa^{K*}\zeta^*\left(1+\frac{3}{16}c_2\right) + \frac{D-1}{D}c_2+\frac{\partial n^{*2}\chi}{\partial n^*}S_D\frac{3(D-1)}{8D^2(D+2)}(1+\alpha)\left(-\alpha(1-\alpha)+\frac{1}{6}(3\alpha^2-3\alpha+10+2D)c_2\right)\right],$$

$$\eta^K = \eta_0\left(\nu_\eta^*-\frac{1}{2}\zeta^*\right)^{-1}\left(1+\frac{S_D}{4D(D+2)}n^*\chi(\alpha+1)(3\alpha-1)\right),$$

$$a_2^{\vec{\nabla}u} = \frac{\eta_0}{nk_BT}\frac{1}{4D(D+2)}(\nu_\gamma^*-D\zeta^*)^{-1}, \tag{99}$$

where $\kappa^{K*}=\kappa^K/\kappa_0$ and

$$\omega^* = (1+\alpha)\left\{(5\alpha-1)(1-\alpha)(1+\alpha) - \tfrac{1}{6}c_2[15\alpha^3-3\alpha^2+3(4D+15)\alpha-(20D+1)]\right\}. \tag{100}$$

(Note that the factor of c_2 coming from $\xi^{(0)}$ is, in accord with the present approximations, retained in the numerators but not the denominators in these expressions.) Before proceeding, it is worth remarking that the transport coefficients derived from the Boltzmann equation (which are only applicable at low density) correspond to the kinetic contributions given in Eq. (99) provided that all explicit factors of χ are set to zero and the implict factors that occur in the definition of the quantities $\nu_\eta^*, \nu_T^*, \nu_\gamma^*$, and ζ^* are set to one. Then, up to the systematically neglected terms of c_2 in the denominators of these expressions, Eq. (99) agree with the previous results of Brey and Cubero [4] obtained directly from the Boltzmann equation.

The collisional parts of the transport coefficients are

$$\eta^C = \eta^K \frac{S_D}{D(D+2)} n^* \chi\left(\frac{1+\alpha}{2}\right) + \frac{D}{D+2}\gamma,$$

$$\gamma = \eta_0 \frac{4S_D^2}{\pi(D+2)D^2}\left(\frac{1+\alpha}{2}\right) n^{*2}\chi\left(1-\frac{1}{16}c_2\right),$$

$$\mu^C = \frac{3S_D}{2D(D+2)} n^* \chi\left(\frac{1+\alpha}{2}\right)\mu^K,$$

$$\kappa^C = \frac{3S_D}{2D(D+2)} n^* \chi\left(\frac{1+\alpha}{2}\right)\kappa^K + \kappa_0(1+\alpha)\frac{2S_D^2(D-1)}{\pi D^2(D+2)^2} n^{*2}\chi\left(1-\frac{7}{16}c_2\right), \quad (101)$$

and the first order correction to the cooling rate is

$$\xi^{(1)} = \xi_0[f_1] + \xi_1[f_0],$$

$$\xi_0[f_1] = -a_2^{\vec{\nabla}u}(\vec{\nabla}\cdot\vec{u})\left(\frac{3}{2}nk_BT\right)\frac{nk_BT}{\eta_0}\frac{D+2}{64}(1-\alpha^2)\chi,$$

$$\xi_1[f_0] = (\vec{\nabla}\cdot\vec{u})\left(\frac{3}{2}nk_BT\right)\frac{S_D}{4D}n^*\chi(1-\alpha^2). \quad (102)$$

Taking into account that the quantities $\zeta = -(2/3nk_BT)\xi$, $c^* = 2c_2$, and $c_D = \frac{1}{2}a_2^{\vec{\nabla}u}$ are used in Ref. [6], it is easy to verify that the present expressions agree for the special case of $D=3$ with those of Ref. [6] up to terms of order $c^* a_{s_0}^{(\gamma)}$. To confirm the expression for $a_2^{\vec{\nabla}u}$ requires that one note that λ^* of Ref. [6] is related to the quantities here by

$$\frac{32}{3}\chi n^* \pi\lambda^* = \chi n^* S_D\omega^* - 2\frac{16D(D+2)}{nk_BT}p^{(0)c}\left(\frac{1}{3}-\alpha\right)c_2$$
$$= \chi n^* S_D(1+\alpha)\Bigg((5\alpha-1)(1-\alpha)(1+\alpha) - \frac{1}{6}c_2[15\alpha^3 - 3\alpha^2 - 3\alpha(12D+17) - 4D + 31]\Bigg), \quad (103)$$

so that the present result can be written as

$$a_2^{\vec{\nabla}u} = 2\frac{\eta_0}{nk_BT}\left(\frac{1}{2}\nu_\gamma^* - \frac{D}{2}\zeta^*\right)^{-1}\left[\frac{2}{3D(D+2)}\chi n^*\pi\lambda^* + \frac{1}{nk_BT}p^{(0)c}\left(\frac{1}{3}-\alpha\right)c_2\right]. \quad (104)$$

Aside from the terms explicitly neglected here, this differs from Ref. [6] in three ways, the coefficient of ζ^* in the denominator, the value of ν_γ^* and the coefficient of c_2 in the numerator are all different from those given in Ref. [6]. It has been confirmed [34] that the expressions given here are correct.

V. CONCLUSIONS

A normal solution to the Enskog approximation for a hard-sphere gas with energy loss has been determined using the Chapman-Enskog procedure to first order in the gradients, and the transport properties given to second order thus specifying the Navier-Stokes hydrodynamic description. The zeroth order distribution function, which describes the homogeneous cooling state, was expanded about equilibrium, Eq. (26), and the equations for the coefficients given. The required collision integrals were expressed in terms of a generating function allowing for evaluation using symbolic mathematical packages and the explicit form of the required integrals needed to determine the first correction to the Gaussian approximation were given explicitly in the form of one-dimensional integrals. The expressions for the transport properties have similarly been reduced to simple quadratures for the standard lowest-Sonine approximation. The Navier-Stokes equations for such a system thus take the form

$$\frac{\partial}{\partial t}n + \vec{\nabla}\cdot(\vec{u}n) = 0,$$

$$\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u} + \frac{1}{\rho}\vec{\nabla}\cdot\vec{P} = 0,$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)T + \frac{2}{Dnk_B}[\vec{P}:\vec{\nabla}\vec{u} + \vec{\nabla}\cdot\vec{q}] = \xi_0 + \xi_1\vec{\nabla}\cdot\vec{u}, \quad (105)$$

with pressure tensor

$$P_{ij} = p^{(0)}\delta_{ij} - \eta\left(\partial_i u_j + \partial_j u_i - \frac{2}{D}\delta_{ij}(\vec{\nabla}\cdot\vec{u})\right) - \gamma\delta_{ij}(\vec{\nabla}\cdot\vec{u}) \quad (106)$$

and heat-flux vector

$$\vec{q}(\vec{r},t) = -\mu\vec{\nabla}\rho - \kappa\vec{\nabla}T, \quad (107)$$

where μ represents a transport coefficient not present when the collision conserve energy. Equations (80)–(85) determine the kinetic parts of the transport coefficients and Eqs. (88) and (89) determine their collisional parts. The pressure, $p^{(0)}$, is given in Eq. (43) and the source term in the temperature equation, which accounts for the cooling, is given by Eq. (48) and Eq. (90). Finally, as a simple application, the transport properties for a granular fluid in D dimensions were given and previous results for low density fluids and for dense fluids in the limit $D=3$ recovered.

It was argued that the polynomial expansions used to solve the various linear integral equations should not be treated as independent and that this implied the systematic

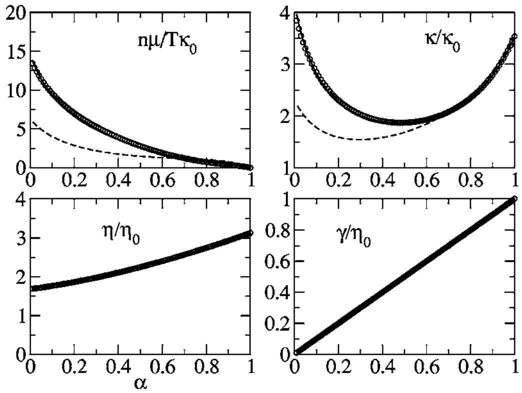

FIG. 1. The four transport coefficients for a simple granular fluid in three dimensions at reduced density $n\sigma^3=0.5$. The transport coefficients are shown in dimensionless form as indicated by the labeling of each figure. The lines are the results of Ref. [6], the circles are the results of this paper and the broken line is the Gaussian approximation.

neglect, at lowest order, of mixed terms coming from the two expansions (i.e., that for f_0 and that for the coefficients in the expression of f_1). One question which has been left unanswered is whether we can judge the effect of this approximation compared to what would happen if all such terms were included. To answer this, we show in Fig. 1 the four transport coefficients for a simple granular fluid in three dimensions at a moderately high density of $n\sigma^3=0.5$. The values are calculated based on the expressions of Ref. [6] which include all contributions in c_2, those given in the preceding section and the results of the Gaussian approximation, $c_2=0$. All three approximations are in agreement for the shear and bulk viscosity, but the Gaussian approximation gives quantitatively poor results for thermal conductivity and the transport coefficient μ. On the other hand, the expressions given here are in good agreement with the full expressions for the entire range of the coefficient of restitution thus giving some justification for the ordering of terms used here.

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APPENDIX A: EXPANSION OF THE COLLISION OPERATOR

The collision operator is

$$J[f,f] = -\int dx_2 \bar{T}_-(12)\chi(\vec{q}_1,\vec{q}_2;[n]) f(\vec{q}_1,\vec{v}_1;t) f(\vec{q}_2,\vec{v}_2;t), \tag{A1}$$

where $\chi(\vec{q}_1,\vec{q}_2;[n])$ is the local equilibrium pair distribution function which is in general a functional of the local density and we write binary collision operator $\bar{T}_-(12)$ as

$$\bar{T}_-(12) = \delta(q_{12}-\sigma)\bar{T}'_-(12),$$

$$\bar{T}'_-(12) = \left(\sum_a J_a(\vec{v}_1,\vec{v}_2)(\hat{b}_a)^{-1} K_a(\hat{q}_{12}\cdot\vec{v}_{12}) - 1\right) \times \Theta(-\vec{v}_{12}\cdot\hat{q}_{12})\delta(q_{12}-\sigma)\vec{v}_{12}\cdot\hat{q}_{12}. \tag{A2}$$

Integrating over the argument of the delta function gives

$$J[f,f] = -\sigma^{D-1}\int d\vec{v}_2 d\hat{q}_{12}\bar{T}'_-(12)\chi(\vec{q}_1,\vec{q}_1-\sigma\hat{q}_{12};[n]) \times f(\vec{q}_1,\vec{v}_1;t) f(\vec{q}_1-\sigma\hat{q}_{12},\vec{v}_2;t). \tag{A3}$$

A gradient expansion of the nonlocal terms requires first an expansion of the one-body distribution,

$$f(\vec{q}_1-\sigma\hat{q}_{12},\vec{v}_2;t) = f(\vec{q}_1,\vec{v}_2;t) - \sigma\hat{q}_{12}\cdot\vec{\nabla}_1 f(\vec{q}_1,\vec{v}_2;t) + \cdots . \tag{A4}$$

For a normal solution, this becomes

$$f(\vec{q}_1-\sigma\hat{q}_{12},\vec{v}_2;t) = f(\vec{q}_1,\vec{v}_2;t) - \sigma\hat{q}_{12}\sum_i [\vec{\nabla}_1\psi_i(\vec{q}_1)] \times \frac{\delta}{\delta\psi_i(\vec{q}_1)} f(\vec{q}_1,\vec{v}_2;t) + \cdots \tag{A5}$$

and of the nonlocal dependence on the density of the pair distribution function

$$\begin{aligned}\chi(\vec{q}_1,\vec{q}_1-\sigma\hat{q}_{12};[n]) &= \chi_0(\sigma;n(\vec{q}_1)) + \int d\vec{r}[n(\vec{r})-n(\vec{q}_1)] \times \left(\frac{\delta}{\delta n(\vec{r})}\chi(\vec{q}_1,\vec{q}_1-\sigma\hat{q}_{12};[n])\right)_{n(\vec{q}_1)} + \cdots \\ &= \chi_0(\sigma;n(\vec{q}_1)) + [\vec{\nabla}_1 n(\vec{q}_1)]\int d\vec{r}(\vec{r}-\vec{q}_1) \times \left(\frac{\delta}{\delta n(\vec{r})}\chi(\vec{q}_1,\vec{q}_1-\sigma\hat{q}_{12};[n])\right)_{n(\vec{q}_1)} + \cdots\end{aligned} \tag{A6}$$

which is accurate up to first order in the gradients. For a single-component system, it can be shown [35] that the second term reduces to a simple derivative giving

$$\chi(\vec{q}_1,\vec{q}_1-\sigma\hat{q}_{12};[n]) = \chi_0(\sigma;n(\vec{q}_1)) - \frac{1}{2}[\sigma\hat{q}_{12}\cdot\vec{\nabla}_1 n(\vec{q}_1)] \times \left|\frac{\partial\chi_0(\sigma;n)}{\partial n}\right|_{n(\vec{q}_1)} + \cdots . \tag{A7}$$

Using these results, we can write

$$J[f,f] = J_0[f,f] + J_1[f,f] + \cdots \tag{A8}$$

with

$$J_0[f,f] = -\chi_0(\sigma;n(\vec{q}_1))\int dx_2\bar{T}_-(12) f(\vec{q}_1,\vec{v}_1;t) f(\vec{q}_1,\vec{v}_2;t) \tag{A9}$$

which is, aside from the prefactor of $\chi_0(\sigma;n(\vec{q}_1))$, the Boltzmann collision operator. The second order term is

$$J_1[f,f] = \sum_i [\vec{\nabla}_1 \psi_i(\vec{q}_1)] \cdot \sigma^D \chi_0(\sigma;n(\vec{q}_1)) \times \int d\vec{v}_2 d\hat{q}_{12} \hat{q}_{12} \bar{T}'_-(12) f(\vec{q}_1,\vec{v}_1;t) \frac{\delta}{\delta\psi_i(\vec{q}_1)} f(\vec{q}_1,\vec{v}_2;t) + [\vec{\nabla}_1 n(\vec{q}_1)] \frac{1}{2} \sigma^D \frac{\partial \chi_0(\sigma;n(\vec{q}_1))}{\partial n(\vec{q}_1)} \times \int d\vec{v}_2 d\hat{q}_{12} \hat{q}_{12} \bar{T}'_-(12) f(\vec{q}_1,\vec{v}_1;t) f(\vec{q}_1,\vec{v}_2;t) \quad (A10)$$

which we write more compactly as

$$J_1[f,f] = \sum_i [\vec{\nabla}_1 \psi_i(\vec{q}_1)] \left(\vec{J}_1 \left[f, \frac{\delta}{\delta\psi_i(\vec{q}_1)} f \right] + \delta_{in} \frac{\partial \ln \chi_0(\sigma;n(\vec{q}_1))}{\partial n(\vec{q}_1)} \vec{J}_1[f,f] \right) \quad (A11)$$

with

$$\vec{J}_1[f,g] = \int d\vec{v}_2 d\hat{q}_{12} \hat{q}_{12} \bar{T}'_-(12) f(\vec{q}_1,\vec{v}_1;t) g(\vec{q}_1,\vec{v}_2;t). \quad (A12)$$

APPENDIX B: EXPANSION OF THE FLUXES AND SOURCES

The expansion of the fluxes and sources is very similar to that of the collision operator described in Appendix A so only a few details will be given here.

1. Pressure tensor

The exact expression for the pressure tensor is $\overleftrightarrow{P} = \overleftrightarrow{P}^K + \overleftrightarrow{P}^C$ with

$$\overleftrightarrow{P}^K(\vec{r},t|f) = m \int d\vec{v}_1 f(\vec{r},\vec{v}_1,t) \vec{V}_1 \vec{V}_1, \quad (B1)$$

and the collisional contribution which can be written as

$$\overleftrightarrow{P}^C(\vec{r},t|f) = -\frac{m}{4} \sigma^D \sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q} \hat{q}\hat{q} (\hat{q} \cdot \vec{v}_{12}) \Theta(-\hat{q} \cdot \vec{v}_{12}) K_a(\hat{q} \cdot \vec{v}_{12}) \int_0^1 dy \chi(\vec{r} + (1-y)\sigma\hat{q}, \vec{r} - y\sigma\hat{q};[n]) \times f(\vec{r} + (1-y)\sigma\hat{q},\vec{v}_1;t) f(\vec{r} - y\sigma\hat{q},\vec{v}_2;t) \left(-\vec{v}_{12} \cdot \hat{q} - \mathrm{sgn}(\vec{v}_{12} \cdot \hat{q}) \sqrt{(\vec{v}_{12} \cdot \hat{q})^2 - \frac{4}{m} \Delta_a(\hat{q} \cdot \vec{v}_{12})} \right). \quad (B2)$$

Clearly, the expansion of kinetic part is simply due to the expansion of the distribution function itself

$$\overleftrightarrow{P}^K(\vec{r},t|f) = \sum_{i=0} \epsilon^i \overleftrightarrow{P}^{K(i)}(\vec{r},t) \quad (B3)$$

with

$$\overleftrightarrow{P}^{K(i)}(\vec{r},t) = \overleftrightarrow{P}^K(\vec{r},t|f_i).$$

The zeroth order contribution is based on f_0 which is homogeneous so

$$P_{ij}^{K(0)}(\vec{r},t|f) = m \frac{1}{D} \delta_{ij} \int d\vec{V} f_0(\vec{V}|\psi_t) V^2 = n k_B T \delta_{ij} \quad (B4)$$

from the definition of the temperature. The first order contribution is

$$P_{ij}^{K(1)}(\vec{r},t|f) = m \int d\vec{V} f_1(\vec{V}|\psi_t) V_i V_j \quad (B5)$$

and comparison to the definition of the first order term Eq. (55) shows that the only contribution is due to the shear term

$$P_{ij}^{K(1)}(\vec{r},t|f) = m \int d\vec{V} f_0(x_1) \left[\sqrt{\frac{D}{D-1}} A^{(\partial u)}(\vec{V}) \left(V_l V_k - \frac{1}{D} \delta_{lk} V^2 \right) \left(\partial_l u_k + \partial_l u_k - \frac{2}{D} \delta_{lk} \vec{\nabla} \cdot \vec{u} \right) \right] V_i V_j = m \sqrt{\frac{D}{D-1}} \left(\partial_l u_k + \partial_l u_k - \frac{2}{D} \delta_{lk} \vec{\nabla} \cdot \vec{u} \right) \frac{1}{D(D+2)} (\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) \int d\vec{V} f_0(x_1) A^{(\partial u)}(\vec{V}) V^4 \quad (B6)$$

and using the expansion of $A^{(\partial u)}$ in associated Laguerre polynomials and their orthogonality relation gives

$$P_{ij}^{K(1)}(\vec{r},t|f) = mnS_D\sqrt{\frac{D}{D-1}}\left(\partial_i u_j + \partial_j u_i - \frac{2}{D}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right)\frac{1}{D(D+2)}\sum_{ij} a_i^{\partial u} c_j\left(\frac{2k_BT}{m}\right)^2\pi^{-D/2}$$
$$\times\int_0^\infty \exp(-x)L_i^{(D+2)/2}(x)L_j^{(D-2)/2}(x)x^{(D+2)/2}dx$$
$$= mnS_D\pi^{-D/2}\sqrt{\frac{D}{D-1}}\left(\partial_i u_j + \partial_j u_i - \frac{2}{D}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right)\frac{1}{D(D+2)}\sum_{ij}\frac{\Gamma\left(\frac{D}{2}+i+2\right)}{\Gamma(i+1)}a_i^{\partial u}\left(\frac{2k_BT}{m}\right)^2(c_i - 2c_{i+1} + c_{i+2})$$
$$= nk_BT\left[2a_0^{\partial u}\left(\frac{k_BT}{m}\right)(1+c_2)\sqrt{\frac{D}{D-1}}\right]\left(\partial_i u_j + \partial_j u_i - \frac{2}{D}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right) \tag{B7}$$

and in the present approximation, we drop the term c_2.

The collisional part of the stress tensor will have terms arising from the expansion of the distribution as well as gradient terms arising from its nonlocality. The former gives a first order contribution of

$$\overleftrightarrow{P}^{C(11)}(\vec{r},t|f) = -\frac{m}{4}\sigma^D\chi_0(\sigma;n(\vec{r}))\sum_a\int d\vec{v}_1 d\vec{v}_2 d\hat{q}\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}_{12}\cdot\vec{v}_{12})[f_0(\vec{r},\vec{v}_1;t)f_1(\vec{r},\vec{v}_2;t) + f_1(\vec{r},\vec{v}_1;t)f_0(\vec{r},\vec{v}_2;t)]$$
$$\times\left(-\vec{v}_{12}\cdot\hat{q}_{12} - \mathrm{sgn}(\vec{v}_{12}\cdot\hat{q}_{12})\sqrt{(\vec{v}_{12}\cdot\hat{q}_{12})^2 - \frac{4}{m}\Delta_a(\hat{q}_{12}\cdot\vec{v}_{12})}\right) \tag{B8}$$

while, using the results of Appendix A, the latter gives two terms

$$\overleftrightarrow{P}^{C(12)}(\vec{r},t) = -\frac{m}{4}\sigma^{D+1}\chi_0(\sigma;n(\vec{r}))\sum_a\int d\vec{v}_1 d\vec{v}_2 d\hat{q}\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})\frac{1}{2}\hat{q}\cdot\{[\vec{\nabla}f_0(\vec{r},\vec{v}_1;t)]f_0(\vec{r},\vec{v}_2;t) - f_0(\vec{r},\vec{v}_1;t)$$
$$\times[\vec{\nabla}f_0(\vec{r},\vec{v}_2;t)]\}\left(-\vec{v}_{12}\cdot\hat{q} - \mathrm{sgn}(\vec{v}_{12}\cdot\hat{q})\sqrt{(\vec{v}_{12}\cdot\hat{q})^2 - \frac{4}{m}\Delta_a(\hat{q}\cdot\vec{v}_{12})}\right) \tag{B9}$$

and

$$\overleftrightarrow{P}^{C(13)}(\vec{r},t) = \frac{m}{4}\frac{1}{2}\sigma^{D+1}\frac{\partial\chi_0(\sigma;n(\vec{r}))}{\partial n(\vec{r})}[\vec{\nabla}_1 n(\vec{r})]\cdot\sum_a\int d\vec{v}_1 d\vec{v}_2 d\hat{q}\hat{q}\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})f_0(\vec{r},\vec{v}_1;t)f_0(\vec{r},\vec{v}_2;t)$$
$$\times\left(-\vec{v}_{12}\cdot\hat{q} - \mathrm{sgn}(\vec{v}_{12}\cdot\hat{q})\sqrt{(\vec{v}_{12}\cdot\hat{q})^2 - \frac{4}{m}\Delta_a(\hat{q}\cdot\vec{v}_{12})}\right) \tag{B10}$$

so that $\overleftrightarrow{P}^{C(1)} = \overleftrightarrow{P}^{C(11)} + \overleftrightarrow{P}^{C(12)} + \overleftrightarrow{P}^{C(13)}$. However, it is seen that $\overleftrightarrow{P}^{C(13)}(\vec{r},t|f)=0$ because of the integral is a vector but there are no zero-order vectors available from which to construct it. For similar reasons, $\overleftrightarrow{P}^{C(12)}(\vec{r},t)$ can be simplified to

$$\overleftrightarrow{P}^{C(12)}(\vec{r},t) = -\frac{m}{8}\sigma^{D+1}\chi_0(\sigma;n(\vec{r}))(\partial_i u_j)\sum_a\int d\vec{v}_1 d\vec{v}_2 d\hat{q}\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})\hat{q}_j\frac{m}{k_BT}\left(V_{1i}\frac{\partial}{\partial z_1}f_0(\vec{r},\vec{v}_1;t)f_0(\vec{r},\vec{v}_2;t)\right.$$
$$\left. - f_0(\vec{r},\vec{v}_1;t)V_{2i}\frac{\partial}{\partial z_2}f_0(\vec{r},\vec{v}_2;t)\right)\left(-\vec{v}_{12}\cdot\hat{q} - \mathrm{sgn}(\vec{v}_{12}\cdot\hat{q})\sqrt{(\vec{v}_{12}\cdot\hat{q})^2 - \frac{4}{m}\Delta_a(\hat{q}\cdot\vec{v}_{12})}\right). \tag{B11}$$

where $z \equiv (m/2k_BT)V^2$.

2. Heat flux vector

The expansion of the kinetic part of the heat flux vector is treated analogous to that of the pressure tensor. It is given by

$$q_i^K(\vec{r},t|f) = \frac{1}{2}m\int d\vec{v}f(\vec{r},\vec{v},t)V_iV^2, \tag{B12}$$

and the zeroth order contribution vanishes by rotational symmetry. The first order contribution is

$$q_i^{K(1)}(\vec{r},t|f) = \frac{1}{2}m\int d\vec{v}f_1(\vec{r},\vec{v},t)V_iV^2 = \frac{1}{2}m\int d\vec{v}f_0(\vec{r},\vec{v},t)[A^{(n)}(\vec{V})V_j\partial_j n + A^{(T)}(\vec{V})V_j\partial_j T]V_iV^2$$
$$= \frac{1}{2D}m\int d\vec{v}f_0(\vec{r},\vec{v},t)[A^{(n)}(\vec{V})V^4\partial_i n + A^{(T)}(\vec{V})V^4\partial_i T], \tag{B13}$$

where the vanishing contributions have been dropped. For both the density and temperature couplings, the important integral is

$$\frac{1}{2D}m\int d\vec{v}f_0(\vec{r},\vec{v},t)A^{(\alpha)}(\vec{V})V^4$$
$$= \frac{1}{2}S_D\frac{1}{2D}mn\left(\frac{2k_BT}{m}\right)^2\sum_{ij}a_i^{(\alpha)}c_j\pi^{-D/2}\int_0^\infty e^{-x}L_i^{D/2}(x)L_j^{(D-2)/2}(x)x^{(D+2)/2}dx$$
$$= \frac{1}{2}S_D\frac{1}{2D}mn\left(\frac{2k_BT}{m}\right)^2\sum_{ij}a_i^{(\alpha)}c_j\pi^{-D/2}\int_0^\infty e^{-x}(L_i^{(D+2)/2}(x) - L_{i-1}^{(D+2)/2}(x))(L_j^{(D+2)/2}(x) - 2L_{j-1}^{(D+2)/2}(x) + L_{j-2}^{(D+2)/2}(x))x^{(D+2)/2}dx$$
$$= \frac{1}{2D}mn\left(\frac{2k_BT}{m}\right)^2\sum_k a_k^{(\alpha)}\frac{\Gamma\left(\frac{D}{2}+k+1\right)}{\Gamma(k+1)\Gamma(D/2)}\left[\left(\frac{1}{2}D+1+3k\right)c_k - 2\left(\frac{1}{2}D+1+\frac{3}{2}k\right)c_{k+1} + \left(\frac{1}{2}D+1+k\right)c_{k+2} - kc_{k-1}\right]$$
$$= nk_BT\left(\frac{k_BT}{m}\right)\frac{D+2}{2}a_1^{(\alpha)}\left(-1-(D+5)c_2+\frac{D+4}{2}c_3\right)+\cdots\,. \tag{B14}$$

So

$$\mu^K = \left[nk_BT\left(\frac{k_BT}{m}\right)\frac{D+2}{2}\right]a_1^\rho\left(1+(D+5)c_2-\frac{D+4}{2}c_3\right),$$
$$\kappa^K = \left[nk_BT\left(\frac{k_BT}{m}\right)\frac{D+2}{2}\right]a_1^T\left(1+(D+5)c_2-\frac{D+4}{2}c_3\right) \tag{B15}$$

which gives the expressions in the text if the terms c_2 and c_3 are neglected.

The collisional part is

$$\vec{q}^C(\vec{r},t) = -\frac{m}{4V}\sigma^D\sum_a\int d\vec{v}_1d\vec{v}_2d\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})\int_0^1 dy\chi(\vec{r}+(1-y)\sigma\hat{q},\vec{r}-y\sigma\hat{q};[n])f(\vec{r}+(1-y)\sigma\hat{q},\vec{v}_1;t)$$
$$\times f(\vec{r}-y\sigma\hat{q},\vec{v}_2;t)\frac{1}{2}(\vec{V}_1+\vec{V}_2)\cdot\hat{q}\left(-\vec{v}_{12}\cdot\hat{q}-\mathrm{sgn}(\vec{v}_{12}\cdot\hat{q})\sqrt{(\vec{v}_{12}\cdot\hat{q})^2-\frac{4}{m}\Delta_a(\hat{q}\cdot\vec{v}_{12})}\right) \tag{B16}$$

which gives at zeroth order

$$\vec{q}^{C(0)}(\vec{r},t) = -\frac{m}{4V}\sigma^D\chi_0(\sigma;n(\vec{r}))\sum_a\int d\vec{v}_1d\vec{v}_2d\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})f_0(\vec{r},\vec{v}_1;t)f_0(\vec{r},\vec{v}_2;t)$$
$$\times\frac{1}{2}(\vec{V}_1+\vec{V}_2)\cdot\hat{q}\left(-\vec{v}_{12}\cdot\hat{q}-\mathrm{sgn}(\vec{v}_{12}\cdot\hat{q})\sqrt{(\vec{v}_{12}\cdot\hat{q})^2-\frac{4}{m}\Delta_a(\hat{q}\cdot\vec{v}_{12})}\right), \tag{B17}$$

and at first order three contributions analogous to those of the pressure tensor

$$\vec{q}^{C(11)}(\vec{r},t) = -\frac{m}{4V}\sigma^D\chi_0(\sigma;n(\vec{r}))\sum_a\int d\vec{v}_1d\vec{v}_2d\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})[f_0(\vec{r},\vec{v}_1;t)f_1(\vec{r},\vec{v}_2;t)+f_1(\vec{r},\vec{v}_1;t)f_0(\vec{r},\vec{v}_2;t)]$$
$$\times\frac{1}{2}(\vec{V}_1+\vec{V}_2)\cdot\hat{q}\left(-\vec{v}_{12}\cdot\hat{q}-\mathrm{sgn}(\vec{v}_{12}\cdot\hat{q})\sqrt{(\vec{v}_{12}\cdot\hat{q})^2-\frac{4}{m}\Delta_a(\hat{q}\cdot\vec{v}_{12})}\right), \tag{B18}$$

and

$$\vec{q}^{C(12)}(\vec{r},t) = -\frac{m}{4V}\sigma^{D+1}\chi_0(\sigma;n(\vec{r}))\sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})\frac{1}{2}\hat{q}\{[\vec{\nabla}f_0(\vec{r},\vec{v}_1;t)]f_0(\vec{r},\vec{v}_2;t) - f_0(\vec{r},\vec{v}_1;t) \times[\vec{\nabla}f_0(\vec{r},\vec{v}_2;t)]\}\frac{1}{2}(\vec{V}_1+\vec{V}_2)\cdot\hat{q}\left(-\vec{v}_{12}\cdot\hat{q} - \text{sgn}(\vec{v}_{12}\cdot\hat{q})\sqrt{(\vec{v}_{12}\cdot\hat{q})^2 - \frac{4}{m}\Delta_a(\hat{q}\cdot\vec{v}_{12})}\right) \quad \text{(B19)}$$

and

$$\vec{q}^{C(13)}(\vec{r},t) = \frac{m}{4V}\frac{1}{2}\sigma^{D+1}\frac{\partial\chi_0(\sigma;n(\vec{r}))}{\partial n(\vec{r})}[\vec{\nabla}n(\vec{r})]\sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q}\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})f_0(\vec{r},\vec{v}_1;t)f_0(\vec{r},\vec{v}_2;t) \times\frac{1}{2}(\vec{V}_1+\vec{V}_2)\cdot\hat{q}\left(-\vec{v}_{12}\cdot\hat{q} - \text{sgn}(\vec{v}_{12}\cdot\hat{q})\sqrt{(\vec{v}_{12}\cdot\hat{q})^2 - \frac{4}{m}\Delta_a(\hat{q}\cdot\vec{v}_{12})}\right). \quad \text{(B20)}$$

Now, the zeroth order heat flux vector vanishes by rotational symmetry of the homogeneous system. Only the vector parts of f_1, i.e., those proportional to $\vec{\nabla}n$ and $\vec{\nabla}T$, can contribute to the first order contribution $\vec{q}^{C(11)}$. Similarly, the second term $\vec{q}^{C(12)}$ can only depend on those gradients but obviously the contribution proportional to $\vec{\nabla}n$ vanishes. Finally, the third contribution vanishes as the integrand is odd in the total momentum.

3. Heat source

The heat source is

$$\xi(\vec{r},t) = \frac{1}{2}\sigma^{D-1}\sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q}(\hat{q}\cdot\vec{v}_{12}) \times\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})\Delta_a(\hat{q}\cdot\vec{v}_{12})\chi(\vec{q}_1,\vec{q}_2;[n]) \times f(\vec{r},\vec{v}_1;t)f(\vec{r}-\sigma\hat{q},\vec{v}_2;t) \quad \text{(B21)}$$

so at zeroth order

$$\xi_0(\vec{r},t) = \frac{1}{2}\sigma^{D-1}\chi_0(\sigma;n(\vec{r}))\sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q}(\hat{q}\cdot\vec{v}_{12}) \times\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})\Delta_a(\hat{q}\cdot\vec{v}_{12}) \times f_0(\vec{r},\vec{v}_1;t)f_0(\vec{r},\vec{v}_2;t). \quad \text{(B22)}$$

There are, as usual, three first order contributions but as in the main text, we separate these into $\xi_1(\vec{r},t)=\xi_0[f_1]+\xi_{11}(\vec{r},t)$ with

$$\xi_0[g] = \frac{1}{2}\sigma^{D-1}\chi_0(\sigma;n(\vec{r}))\sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12}) \times K_a(\hat{q}\cdot\vec{v}_{12})\Delta_a(\hat{q}\cdot\vec{v}_{12})[f_0(\vec{r},\vec{v}_1;t)g(\vec{r},\vec{v}_2;t) + g(\vec{r},\vec{v}_1;t)f_0(\vec{r},\vec{v}_2;t)] \quad \text{(B23)}$$

and

$$\xi_{11}(\vec{r},t) = -\frac{1}{2}\sigma^{D}(\partial_j\psi_i)\sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q}\hat{q}_j(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12}) \times K_a(\hat{q}\cdot\vec{v}_{12})\Delta_a(\hat{q}\cdot\vec{v}_{12})\chi(\vec{q}_1,\vec{q}_2;[n])f_0(\vec{r},\vec{v}_1;t) \times\frac{\partial}{\partial\psi_i}f_0(\vec{r},\vec{v}_2;t), \quad \text{(B24)}$$

and note that the coefficient of $\partial\chi_0(\sigma;n(\vec{r}))/\partial n(\vec{r})$ vanishes by spherical symmetry. The only nonzero contribution to ξ_{11} comes from $\psi_i=\vec{u}$ and the integral must be proportional to the unit tensor so we can write $\xi_{11}(\vec{r},t)=\xi_1^{\vec{\nabla}u}(\vec{r},t)\vec{\nabla}\cdot\vec{u}$ with

$$\xi_1^{\vec{\nabla}u}(\vec{r},t) = -\frac{1}{2}\sigma^{D}\sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12}) \times K_a(\hat{q}\cdot\vec{v}_{12})\Delta_a(\hat{q}\cdot\vec{v}_{12})\chi(\vec{q}_1,\vec{q}_2;[n])f_0(\vec{r},\vec{v}_1;t)\hat{q}_j \times\frac{\partial}{\partial u_j}f_0(\vec{r},\vec{v}_2;t). \quad \text{(B25)}$$

APPENDIX C: GENERATING FUNCTION FOR THE HCS

The definition of the couplings is

$$I_{rs,k} = -n^{-1}A_k^{-1}\int d\vec{v}_1 d2 L_k^{(D-2)/2}\left(\frac{m}{2k_BT}v_1^2\right)\overline{T}_-(12) \times\left(\frac{2k_BT}{m}\right)^{-D} f_M(\vec{v}_1)f_M(\vec{v}_2) \times L_r^{(D-2)/2}\left(\frac{m}{2k_BT}v_1^2\right)L_s^{(D-2)/2}\left(\frac{m}{2k_BT}v_1^2\right) \quad \text{(C1)}$$

which is equivalent to

$$I_{rs,k} = n^{-1}A_k^{-1}\int d\vec{v}_1 d2\left(\frac{2k_BT}{m}\right)^{-D} f_M(\vec{v}_1)f_M(\vec{v}_2) \times L_r^{(D-2)/2}\left(\frac{m}{2k_BT}v_1^2\right)L_s^{(D-2)/2}\left(\frac{m}{2k_BT}v_1^2\right)T_+(12) \times L_k^{(D-2)/2}\left(\frac{m}{2k_BT}v_1^2\right). \tag{C2}$$

The associated Laguerre polynomials are generated by

$$L_n^{\alpha}(x) = \frac{1}{n!}\lim_{z\to 0}\frac{\partial^n}{\partial z^n}\frac{1}{(1-z)^{\alpha+1}}\exp\left(-\frac{xz}{1-z}\right) \tag{C3}$$

so that we have

$$I_{rs,k} = -nA_k^{-1}\left(\frac{m}{2k_BT}\right)^{-1/2}\sigma^{D-1}\frac{1}{r!s!k!}\lim_{z_1\to 0}\lim_{z_2\to 0}\lim_{x\to 0}\frac{\partial^r}{\partial z_1^r}\frac{\partial^s}{\partial z_2^s} \times\frac{\partial^k}{\partial x^k}\left(\sum_a G_a(\Delta_a) - G_0\right)$$

with

$$G_0 = \frac{1}{(1-z_1)^{D/2}}\frac{1}{(1-z_2)^{D/2}}\frac{1}{(1-x)^{D/2}}\pi^{-D}\int d\vec{v}_1 d\vec{v}_2 d\hat{q} \times\exp\left(-\frac{1}{1-z_1}v_1^2 - \frac{1}{1-z_2}v_2^2\right)(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12}) \times\exp\left(-\frac{x}{1-x}v_1^2\right),$$

$$G_a(\Delta) = \frac{1}{(1-z_1)^{D/2}}\frac{1}{(1-z_2)^{D/2}}\frac{1}{(1-x)^{D/2}}\pi^{-D} \times\int d\vec{v}_1 d\vec{v}_2 d\hat{q}K_a^*(\hat{q}\cdot\vec{v}_{12}) \times\exp\left(-\frac{1}{1-z_1}v_1^2 - \frac{1}{1-z_2}v_2^2\right)(\hat{q}\cdot\vec{v}_{12}) \times\Theta(-\hat{q}\cdot\vec{v}_{12})\hat{b}_a\exp\left(-\frac{x}{1-x}v_1^2\right). \tag{C4}$$

To evaluate the second quantity, note that

$$v_1'^2 = V^2 + \vec{V}\cdot\vec{v} - \vec{V}\cdot\hat{q}(\vec{v}\cdot\hat{q} + \mathrm{sgn}(\vec{v}\cdot\hat{q}) \times\sqrt{(\vec{v}\cdot\hat{q})^2 - 2\beta\Delta(\vec{v}\cdot\hat{q})}) + \frac{1}{4}v^2 - \frac{\beta}{2}\Delta(\vec{v}\cdot\hat{q}). \tag{C5}$$

A tedious calculation to complete the square in V gives

$$G_a = \frac{1}{(1-z_1)^{D/2}}\frac{1}{(1-z_2)^{D/2}}\frac{1}{(1-x)^{D/2}}\pi^{-D}\int d\vec{V}d\vec{v}d\hat{q}(\hat{q}\cdot\vec{v})\Theta(-\hat{q}\cdot\vec{v})K_a^*(\hat{q}\cdot\vec{v}_{12})\exp\left(\frac{(2-z_2-z_1)x}{2-x-z_2-z_1+xz_1z_2}\frac{1}{2}\Delta_a^*(\vec{v}\cdot\hat{q})\right) \times\exp\left[-\left(\frac{1}{1-z_1}+\frac{1}{1-z_2}+\frac{x}{1-x}\right)V^2\right]\exp\left(-\frac{1-z_1x}{2-x-z_2-z_1+xz_1z_2}v^2\right) \times\exp\left(\frac{1}{2}\frac{(z_2-z_1)x}{2-x-z_2-z_1+xz_1z_2}[(\vec{v}\cdot\hat{q})^2 + \vec{v}\cdot\hat{q}\,\mathrm{sgn}(\vec{v}\cdot\hat{q})\sqrt{(\vec{v}\cdot\hat{q})^2 - 2\Delta_a^*(\vec{v}\cdot\hat{q})}]\right). \tag{C6}$$

Performing the V integration and the $D-1$ v integrations in directions perpendicular to $\hat{q}$ and finally the $\hat{q}$ integral leave the final result

$$G_a = -\frac{1}{2}\pi^{-1/2}S_D(1-z_1x)^{-(1/2)D}\left(\frac{1-z_1x}{2-x-z_2-z_1+xz_1z_2}\right)^{1/2}\int_0^{\infty}duK_a^*(\sqrt{u})\exp\left(\frac{(2-z_2-z_1)x}{2-x-z_2-z_1+xz_1z_2}\frac{1}{2}\Delta_a^*(\sqrt{u})\right) \times\exp\left[-\frac{1}{2}\left(\frac{2-z_2x-z_1x}{2-x-z_2-z_1+xz_1z_2}u\right)\right]\exp\left(\frac{1}{2}\frac{(z_2-z_1)x}{2-x-z_2-z_1+xz_1z_2}\sqrt{u}\sqrt{u-2\Delta_a^*(\sqrt{u})}\right). \tag{C7}$$

We will also need the generating function for the case that no collision occurs,

$$G_0 = \frac{1}{(1-z_1)^{D/2}}\frac{1}{(1-z_2)^{D/2}}\frac{1}{(1-x)^{D/2}}\pi^{-D}\int d\vec{v}_1 d\vec{v}_2 d\hat{q}\exp\left(-\frac{1}{1-z_1}v_1^2 - \frac{1}{1-z_2}v_2^2\right)\hat{q}\cdot\vec{v}_{12}\Theta(-\hat{q}\cdot\vec{v}_{12})\exp\left(-\frac{x}{1-x}v_1^2\right). \tag{C8}$$

Completing the square and performing the simple Gaussian integrals gives

$$G_0 = -\tfrac{1}{2}\pi^{-1/2}S_D(1-z_1x)^{-(D+1)/2}(2-x-z_2-z_1+xz_1z_2)^{1/2}. \tag{C9}$$

APPENDIX D: EVALUATION OF INTEGRALS

To illustrate the method used to evaluate the many integrals required in this work, consider the quantity $\vec{q}^{C(12)}(\vec{r},t)$ defined in Eq. (B19) and for convenience repeated here,

$$\vec{q}^{C(12)}(\vec{r},t) = -\frac{m}{4}\sigma^{D+1}\chi_0(\sigma;n(\vec{r}))\sum_a \int d\vec{v}_1 d\vec{v}_2 d\hat{q}\hat{q}(\hat{q}\cdot\vec{v}_{12})\Theta(-\hat{q}\cdot\vec{v}_{12})K_a(\hat{q}\cdot\vec{v}_{12})\frac{1}{2}\hat{q}\{[\vec{\nabla}f_0(\vec{r},\vec{v}_1;t)]f_0(\vec{r},\vec{v}_2;t) - f_0(\vec{r},\vec{v}_1;t) \times[\vec{\nabla}f_0(\vec{r},\vec{v}_2;t)]\}\frac{1}{2}(\vec{V}_1+\vec{V}_2)\cdot\hat{q}\left(-\vec{v}_{12}\cdot\hat{q} - \text{sgn}(\vec{v}_{12}\cdot\hat{q})\sqrt{(\vec{v}_{12}\cdot\hat{q})^2 - \frac{4}{m}\Delta_a(\hat{q}\cdot\vec{v}_{12})}\right). \tag{D1}$$

As stated previously, the only nonzero contribution comes through the temperature so we replace

$$\vec{\nabla}f_0 \to (\vec{\nabla}T)\frac{\partial}{\partial T}f_0. \tag{D2}$$

Keeping terms up to linear order in c_2 and defining the quantities

$$z_i = \frac{m}{2k_BT}V_i^2, \quad \vec{V} = \sqrt{\frac{m}{2k_BT}}(\vec{v}_1+\vec{v}_2), \quad \vec{v} = \sqrt{\frac{m}{2k_BT}}(\vec{v}_1-\vec{v}_2), \tag{D3}$$

we find that

$$\begin{aligned}
&\left(\frac{\partial}{\partial T}f^{(0)}(\vec{r},\vec{v}_1;t)\right)f^{(0)}(\vec{r},\vec{v}_2;t) - f^{(0)}(\vec{r},\vec{v}_1;t)\frac{\partial}{\partial T}f^{(0)}(\vec{r},\vec{v}_2;t)\\
&= -\frac{1}{T}\left[\left(z_1\frac{\partial}{\partial z_1}f^{(0)}(\vec{r},\vec{v}_1;t)\right)f^{(0)}(\vec{r},\vec{v}_2;t) - f^{(0)}(\vec{r},\vec{v}_1;t)z_2\frac{\partial}{\partial z_2}f^{(0)}(\vec{r},\vec{v}_2;t)\right]\\
&= -\frac{1}{T}\left[(-z_1+z_2)\left(1+c_2\left\{\left(\frac{1}{4}D^2+\frac{1}{2}D\right)-(D+2)\left(V^2+\frac{1}{4}v^2\right)+\left[V^4+\frac{1}{2}V^2v^2+\frac{1}{16}v^4+(\vec{V}\cdot\vec{v})^2\right]\right\}\right)\right.\\
&\quad\left.+z_1c_2\left(-\frac{1}{2}(D+2)+z_1\right)-z_2c_2\left(-\frac{1}{2}(D+2)+z_2\right)\right]f_M(\vec{r},\vec{v}_1;t)f_M(\vec{r},\vec{v}_2;t)\\
&= \frac{1}{T}2\vec{V}\cdot\vec{v}\left(1+c_2\left\{\frac{1}{4}(D+2)^2-(D+4)\left(V^2+\frac{1}{4}v^2\right)+\left[V^4+\frac{1}{2}V^2v^2+\frac{1}{16}v^4+(\vec{V}\cdot\vec{v})^2\right]\right\}\right)f_M(\vec{r},\vec{v}_1;t)f_M(\vec{r},\vec{v}_2;t).
\end{aligned} \tag{D4}$$

Substituting into the original expression and changing integration variables gives

$$\begin{aligned}
q_i^{C(12)}(\vec{r},t) &= -\frac{m}{4}n^2\sigma^{D+1}\chi_0(\sigma;n(\vec{r}))(\partial_jT)\left(\frac{2k_BT}{m}\right)^{3/2}\pi^{-D}\sum_a\int d\vec{V}dvd\hat{q}\hat{q}_i\hat{q}_j(\hat{q}\cdot\vec{v})\Theta(-\hat{q}\cdot\vec{v})\\
&\quad\times K_a\left(\sqrt{\frac{2k_BT}{m}}\hat{q}\cdot\vec{v}\right)\frac{1}{2}\frac{1}{T}2\vec{V}\cdot\vec{v}\left(1+c_2\left\{\frac{1}{4}(D+2)^2-(D+4)\left(V^2+\frac{1}{4}v^2\right)\right.\right.\\
&\quad\left.\left.+\left[V^4+\frac{1}{2}V^2v^2+\frac{1}{16}v^4+(\vec{V}\cdot\vec{v})^2\right]\right\}\right)e^{-2V^2-(1/2)v^2}\times\frac{1}{2}(\vec{V}\cdot\hat{q})[-\vec{v}\cdot\hat{q}-\text{sgn}(\vec{v}\cdot\hat{q})\sqrt{(\vec{v}\cdot\hat{q})^2-2\Delta_a^*(\hat{q}\cdot\vec{v})}].
\end{aligned} \tag{D5}$$

Taking $\hat{v}$ to be the x direction for the V integral, this becomes

$$\begin{aligned}
q_i^{C(12)}(\vec{r},t) &= -\frac{m}{8}n^2\sigma^{D+1}\chi_0(\sigma;n(\vec{r}))(\partial_jT)\left(\frac{2k_BT}{m}\right)^{3/2}\frac{1}{T}\pi^{-D}\sum_a\int d\vec{V}d\vec{v}d\hat{q}\hat{q}_i\hat{q}_j(\hat{q}\cdot\vec{v})^2\Theta(-\hat{q}\cdot\vec{v})K_a^*(\hat{q}\cdot\vec{v})V_x^2\left\{1+c_2\left[\frac{1}{4}(D+2)^2\right.\right.\\
&\quad\left.\left.-(D+4)\left(V^2+\frac{1}{4}v^2\right)+\left(V^4+\frac{1}{2}V^2v^2+\frac{1}{16}v^4+V_x^2v^2\right)\right]\right\}e^{-2V^2-(1/2)v^2}(-\vec{v}\cdot\hat{q}-\text{sgn}(\vec{v}\cdot\hat{q})\sqrt{(\vec{v}\cdot\hat{q})^2-2\Delta_a^*(\hat{q}\cdot\vec{v})}).
\end{aligned} \tag{D6}$$

The method now is to take $\hat{q}$ to be the x direction in the v integrals and to make the following substitutions:

$$V^2 \to V_x^2 + V_\perp^2, \quad v^2 \to v_x^2 + v_\perp^2, \tag{D7}$$

and to integrate over V_x, $V_\perp$, and $v_\perp$. These integrals are performed by using symbolic manipulation to expand the kernal and to make the replacement

$$\pi^{-D}V_x^l V_\perp^m v_\perp^n \to \frac{1}{\sqrt{2\pi}}\left[2^{-(l+1)/2}\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{l+1}{2}\right)\right]\left(2^{-m/2}\frac{\Gamma\left(\frac{D-1+m}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)}\right)\left(2^{n/2}\frac{\Gamma\left(\frac{D-1+n}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)}\right) \tag{D8}$$

to get

$$\begin{aligned} q_i^{C(12)}(\vec{r},t) &= -\frac{m}{4}n^2\sigma^{D+1}\chi_0(\partial_j T)\left(\frac{k_B T}{m}\right)^{3/2}\frac{1}{T\sqrt{\pi}}\sum_a \int_{-\infty}^0 dv_x d\hat{q}\hat{q}_i\hat{q}_j v_x^2 K_a^*(v_x)\left(\frac{1}{4}+\frac{1}{64}c_2(v_x^4-2v_x^2-9)\right) \\ &\quad \times e^{-(1/2)v_x^2}[-v_x-\mathrm{sgn}(v_x)\sqrt{v_x^2-2\Delta_a^*(v_x)}] \\ &= -\frac{m}{4V}n^2\sigma^{D-1}\chi_0(\partial_i T)\left(\frac{k_B T}{m}\right)^{3/2}\frac{S_D}{D\sqrt{\pi}}\sum_a \int_0^\infty K_a(v_x)v_x^2 e^{-(1/2)v_x^2}\left(\frac{1}{4}+\frac{1}{64}c_2(v_x^4-2v_x^2-9)\right)[\sqrt{v_x^2-2\Delta_a^*(v_x)}+v_x]dv_x. \end{aligned} \tag{D9}$$

We can isolate the contribution due to $\Delta_a^*(v_x)\neq 0$ by subtracting the $\Delta_a^*(v_x)=0$ term and using $\Sigma_a K_a^*(v_x)=1$ to get

$$\begin{aligned} q_i^{C(12)}(\vec{r},t) = -\frac{m}{4}n^2\sigma^{D+1}\chi_0(\partial_i T)\left(\frac{k_B T}{m}\right)^{3/2}\frac{S_D}{DT\sqrt{\pi}}\Bigg[1+\frac{7}{16}c_2+\frac{1}{4}\sum_a \int_0^\infty K_a^*(v_x)v_x^2 e^{-(1/2)v_x^2}\left(1+\frac{1}{16}c_2(v_x^4-2v_x^2-9)\right) \\ \times[\sqrt{v_x^2-2\Delta_a^*(v_x)}-v_x]dv_x\Bigg]. \end{aligned} \tag{D10}$$

The only further complication is that some integrals involve kernals of the form $h(V^2)(\vec{V}\cdot\hat{q})^2(\vec{V}\cdot\vec{v})^2$. These can be handled by using the substitution

$$h(V^2)V_iV_jV_kV_l \to h(V^2)V^4\frac{1}{D^2+2D}(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}). \tag{D11}$$

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